M(atrix) models for M-theory

Or, fun with big matrices. The subject first came up in

de Wit, Hoppe, Nicolai NPB 305 (1988) 545

was completely reinterpreted by

Banks, Fischler, Shenker, Susskind hep-th/9610043

got subsumed (to my mind) in

Itzhaki, Maldacena, Sonnenschein, Yankielowicz hep-th/9802042

was reviewed by

Taylor hep-th/0101126

and may be making a comeback in

Berenstein, Maldacena, Nastase hep-th/0202021

Outline:

1. light-front kinematics
2. light-front M-theory and the BFSS conjecture
3. recovering the spectrum - gravitons, membranes, 5-branes
4. recovering interactions
5. compactification (on tori, mostly)
6. some difficulties with the BFSS limit
7. the IMSY perspective
8. pp waves

Light-front Kinematics

$x^0, x^1, \ldots, x^{D-1}$ Minkowski coordinates

periodically identify $x^{D-1} \approx x^{D-1} + 2\pi R$

Consider the sector of the theory with N units of momentum around the S^1.

$p^{D-1} = N/R$

For a system of free particles

$$H_{particle} = \sqrt{p^2 + m^2} = \sqrt{\left(\frac{N}{R}\right)^2 + |\vec{p}_\perp|^2 + m^2}$$

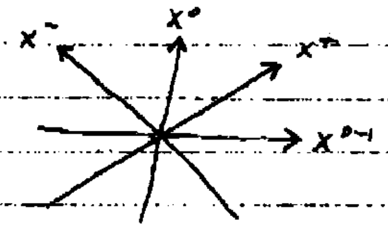
In the "infinite momentum frame" $N \to \infty$ with R fixed, so $p^{D-1} \to \infty$ and

$$H = \frac{N}{R} + \frac{|\vec{p}_\perp|^2 + m^2}{2\,p^{D-1}} + \cdots$$

It's natural to introduce light-front coordinates

$$x^{\pm} = \frac{1}{2}(x^0 \pm x^{D-1})$$

$x^+ = t$ = light-front time coordinate



light-front Hamiltonian

$$H = i\frac{\partial}{\partial t} = i\frac{\partial}{\partial x^+} = i\frac{\partial}{\partial x^0} + i\frac{\partial}{\partial x^{D-1}} = E - p^{D-1} =$$

$$= \frac{|\vec{p}_\perp|^2 + m^2}{2\,p^{D-1}}$$

looks like non-relativistic kinematics with $p^{D-1} \leftrightarrow$ mass ($\to \infty$ as $N \to \infty$)

The underlying $SO(D-1,1)$ Lorentz invariance is broken by the choice of infinite momentum frame, so the light-front Hamiltonian only has a manifest Gallilean symmetry acting on the transverse momenta.

BFSS conjecture

~~Study M-theory in the infinite momentum frame.~~

11-D Planck length ℓ_{11}

periodically identify $x^{11} \approx x^{11} + 2\pi R_{11}$

study sector with N units of momenta on S^1

This is equivalent to IIA string theory with

$$g_s = (R/\ell_{11})^{3/2} \qquad \ell_s = \sqrt{\ell_{11}^3/R}$$

sector with N units of 0-brane charge

So we have a system of N D0-branes. In the IMF the 0-branes are slowly moving in the transverse directions (time dilation!). This means we only need to keep the leading terms in the velocity expansion of the 0-brane effective action. This is uniquely fixed by the symmetries in the IMF frame:

Gallilean invariance in the nine xverse directions

16 supercharges (the ground state in the sector $p^{0-1} = N/R$ is $\frac{1}{2}$-BPS $\Rightarrow$ preserve 16 manifest supercharges in IMF)

$$\Rightarrow S = \frac{1}{2g_s\ell_s} \int dt \; \mathrm{Tr}\left\{ \dot{X}^i \dot{X}^i + \frac{1}{2}[X^i, X^j][X^i, X^j] + \theta^T (i\dot{\theta} - \Gamma_i [X^i, \theta]) \right\}$$

X^i $i = 1, \ldots, 9$ $N \times N$ Hermitian matrices

θ_a $a = 1, \ldots, 16$ Grassmann matrices

This action is the dimensional reduction of $\mathcal{N}=4$ $U(N)$ SYM from 3+1 to 0+1 dimensions (in the gauge $A_0 = 0$). Alternatively, it's the two-derivative truncation of the super-DBI action for 0-branes.

Other stringy degrees of freedom seem to decouple. For example the 0-branes have small velocities ⇒ small accelerations ⇒ won't emit gravitational radiation.

(That doesn't mean gravitational back-reaction is negligable – more later when we talk about IMSY.)

This leads to the BFSS conjecture:

The Yang-Mills quantum mechanics of $N \to \infty$ D0-branes provides a complete description of M theory in the infinite momentum frame.

Recovering the spectrum

As a first test, let's see if we can recover the spectrum of excitations from the 0-brane quantum mechanics. We ought to be able to find

- 11-D SUGRA multiplet $(g_{\mu\nu}, \psi_\mu, C^{(3)}_{\mu\nu\lambda})$
- membranes (coupled to $C^{(3)}$)
- 5-branes (coupled to the dual $C^{(6)}$)

SUGRA multiplet?

First consider a single D0 brane, with

$$S = \frac{1}{2 g_s \ell_s} \int dt \left(\dot{X}^i \dot{X}^i + i \theta^T \dot{\theta} \right)$$

(abelian so no interactions). This is the action for a single free non-relativistic particle moving in the nine xverse dimensions, with mass $m = \frac{1}{g_s \ell_s} =$ expected mass for a D0. The Grassmann coordinates θ_a give 16 real fermion zero modes,

$$\hat{H} = \frac{p^i p^i}{2m} \quad \text{plus} \quad \{\theta_a, \theta_b\} = \delta_{ab}$$

The fermion zero modes give a

$$2^{16/2} = 256$$

dimensional space of states, exactly the

$$128_{\text{bose}} \oplus 128_{\text{ferm}}$$

states in the M-theory SUGRA multiplet.

What about gravitons with more than one unit of p^{11}? To make a graviton with N units of p^{11} bring together N D0 branes in a "threshold bound state".

exactly zero energy, preserves 16 supercharges

relative wavefunction is normalizeable

If such a bound state exists (uniquely!), it will have the right quantum #'s to correspond to a graviton with $p^{11} = N/R$.

arise from c.m. wavefunction

The existence of these threshold bound states has been established, but few properties are known.

Sethi + Stern, 9705046 Porrati + Rozenberg 9708119

Membranes?

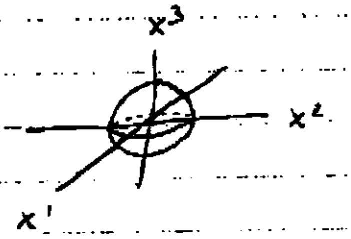
This was the first thing people understood about M(atrix) theory (de Wit Hoppe Nicolai NPB 305 (1988) 545; see also Kabat and Taylor 9711078).

Consider setting

$$X^1 = \frac{2r}{N} J^1 \qquad X^2 = \frac{2r}{N} J^2 \qquad X^3 = \frac{2r}{N} J^3$$

J^i = generators of N-dimensional rep. of $SU(2)$

This describes a spherical M-2-brane of radius r, embedded in $R^3 = \{(x^1, x^2, x^3)\}$.



To see this note that

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = \frac{4r^2}{N^2} J^i J^i = r^2\left(1 - \frac{1}{N^2}\right) \mathbb{1}$$

so that the 0-branes are localized on the surface of a sphere. The coupling to the 3-form $C_{\mu\nu\lambda}$ arises from the interaction term

$$-\int dt\, C_{0ij}\, Tr([X^i, X^j])$$

in the action for N D0 branes: $[X^i, X^j] \neq 0 \Rightarrow$ non-zero membrane charge (at least locally).

5-branes?

M-5-branes which are wrapped around the S^1 are seen as D-4-branes in IIA. These objects can be described in a manner parallel to the M-2 story,

—collection of four matrices with $\varepsilon_{ijkl}\, Tr(X^i X^j X^k X^l) \neq 0$	$\longleftrightarrow$	M-5-brane wrapped on S^1 and extended in the $x^i x^j x^k x^l$ directions

For explicit matrices describing a round spherical wrapped M-5-brane see Castelino, Lee, Taylor 9712105.

It's not clear how to describe an M-5 that isn't wrapped around the S^1... in fact it may not be possible to have an unwrapped M-5 in the IMF.

Fock space?

To describe multiple objects in M(atrix) theory one introduces block-diagonal matrices.

$$X^i = \begin{pmatrix} \boxed{\text{graviton}} & & & 0 \\ & \boxed{\text{graviton}} & & \\ & & \boxed{\text{membrane}} & \\ 0 & & & \ddots \end{pmatrix}$$

The interactions only involve commutators,

$$\mathcal{L}_{int} = \mathrm{Tr}\left(\tfrac{1}{2}[X^i, X^j][X^i, X^j] - \theta^T \Gamma_i [X^i, \theta] \right)$$

so classically the different blocks don't interact with each other. Quantum mechanically, if the objects are mutually BPS, they still don't interact.

Recovering interactions

Suppose we had two objects which weren't mutually BPS. How can we get their interaction potential? For example

$$\mathbf{X}^i = \begin{pmatrix} \text{graviton at } \vec{x}_1 & \\ & \text{graviton at } \vec{x}_2 \end{pmatrix}$$

potential $V(\vec{x}_1 - \vec{x}_2)$?

M(atrix) theory:

Integrate out the off-diagonal blocks to compute an effective potential $V_{\text{matrix}}(\vec{x}_1 - \vec{x}_2)$. You can do this explicitly at one loop,

1^{st} graviton described by $F_1^{0i} = \dot{\mathbf{X}}_1^i$

$F_1^{ij} = i[\mathbf{X}_1^i, \mathbf{X}_1^j]$

2^{nd} graviton described by $F_2^{0i} = \dot{\mathbf{X}}_2^i$

$F_2^{ij} = i[\mathbf{X}_2^i, \mathbf{X}_2^j]$

$$V_{\substack{\text{matrix,}\\ \text{one-loop}}} = -\frac{5}{128}\frac{1}{|\vec{x}_1 - \vec{x}_2|^7}\, \mathrm{STr}\left\{24\left(F_1 \otimes \mathbb{1} - \mathbb{1} \otimes F_2\right)^4 - 6\left(\left(F_1 \otimes \mathbb{1} - \mathbb{1} \otimes F_2\right)^2\right)^2\right\}$$

Linearized SUGRA:

$$V_{\text{SUGRA}} = \underbrace{-\frac{15}{4}\frac{R^2}{|\vec{x}_1 - \vec{x}_2|^7}\, \underset{\uparrow}{T_1}\, \underset{\uparrow}{T_2}}_{\text{graviton exchange}} \underbrace{-\frac{45 R^2}{|\vec{x}_1 - \vec{x}_2|^7}\, \underset{\uparrow}{J_1}\, \underset{\uparrow}{J_2}}_{\text{electric 3-form exchange}} \underbrace{-\frac{45 R^2}{|\vec{x}_1 - \vec{x}_2|^7}\, \underset{\uparrow}{M_1}\, \underset{\uparrow}{M_2}}_{\text{magnetic 3-form exchange}}$$

stress tensors — 3-form currents — 6-form currents

Rather remarkably, with a suitable dictionary relating M(atrix) operators to SUGRA currents, the two potentials are in precise agreement!

Kabat + Taylor, 9712185
Taylor + van Raamsdonk, 9812239

For example define $T^{ij} = \frac{1}{R} \mathrm{Tr}\left(\dot{X}^i \dot{X}^j - [X^i, X^k][X^k, X^j]\right)$

Compactification

M(atrix) theory describes M-theory in a Minkowski background in the infinite momentum frame, in terms of the dynamics of $N \to \infty$ 0-branes in flat space.

Natural guess for compactified M-theory: let the 0-branes move on a compact space.

Unfortunately the 0-brane action in a curved background isn't very well understood. Some progress has been made in compactifying on tori,

$$\underset{\text{on } T^p}{N \text{ D-0-branes}} \xleftrightarrow{\text{T-dual}} \underset{\text{on dual } \tilde{T}^p}{N \text{ D-}p\text{-branes}}$$

giving $(p+1)$-dimensional SYM on $\tilde{T}^p$ (Taylor, 9611042). This makes some properties clear, in particular some of the U-duality groups of toroidally compactified M-theory become manifest. However even compactification on T^p breaks down for large p... most dramatically, on $T^{p=7}$, the dual D-7-branes are codimension 2 and have fixed conical deficit angles that limit $N \leq 24$. No $N \to \infty$ limit is possible.

Difficulties with BFSS limit

The reason M(atrix) theory is hard is that the 0-brane quantum mechanics is strongly coupled. To describe M-theory in the IMF with $R \gg \ell_{11}$ we need

$$N \to \infty \qquad g_{YM}^2 = g_s \ell_s = \left(\frac{R}{\ell_{11}}\right)^{3/2} \cdot \frac{\ell_{11}^{3/2}}{\sqrt{R}} = R \quad \text{fixed}$$

(or even $g_{YM}^2 \to \infty$ to decompactify S^1)

So there's very little control over the 0-brane quantum mechanics, aside from the BPS spectrum. Even the spectacular matching of potentials we saw earlier can be traced to SUSY non-renormalization theorems which only apply at one loop (Sethi, Paban, Stern).

There may be multiloop non-renormalization theorems for some terms in the action.

More concretely, as $N \to \infty$ all states become small perturbations on the (very poorly understood) threshold bound state of N D-0-branes.

IMSY perspective

way to take gravitational back-reaction into account

The BFSS conjecture can be "derived" by studying the near horizon geometry of $N \to \infty$ D-0-branes (Itzhaki et. al, 9802042)

In the BFSS limit one finds that the dual near-horizon geometry is a region of 11-dimensional flat space. Thus BFSS can be regarded as a non-conformal variant of AdS/CFT.

pp waves

Who knows, this may all come back. Berenstein et. al, 0202021 studied M-theory on $AdS_4 \times S^7$, in a sector with $J \to \infty$ units of K.-K. momentum on the S^7. They write down the corresponding M(atrix) model, a U(J) SYM theory. Because of the extra terms in their matrix action, arising from the curved "pp wave" background, the BMN matrix model seems to be much easier to study than BFSS.