1 Classical point particle

Let us consider a classical point particle of mass m moving in a D dimensional target space. For simplicity, we choose the Minkowski space as a target space, therefore the metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. The point particle describes a *worldline* in spacetime and the action describing its motion is given by its proper length

$$S\,[\,X^\mu\,] \;=\; mc\int\sqrt{\,-\,\eta_{\mu\nu}dX^\mu\,dX^\nu}\;=\;mc\int\sqrt{\,-\,\dot{X}^2}\;dt\;. \tag{1}$$

Here $X^\mu(t)$ with $\mu = 0, 1, \ldots, D-1$ are the embedding fields and they describe how the worldline is embedded into the target space.
Under a generic worldline reparametrization $t \to \tilde{t} = f(t)$ the embedding fields are scalars, therefore $S\,[\,X^\mu\,]$ is invariant, as expected from the fact that it is proportional to the proper length of the worldline.
Clearly, since the action in (1) is nonlinear, it is very difficult to work with it. Because of this, we rewrite it by introducing an *auxiliary field* $e(t)$ (*einbein*) as follows

$$S\;[\,X^\mu, e\,]\;=\;\frac{1}{2}\int dt\left(-\,\frac{1}{e}\,\frac{dX^\mu}{dt}\,\frac{dX^\nu}{dt}\,\eta_{\mu\nu}+m^2c^2\,e\right)\;. \tag{2}$$

The new field $e(t)$ is auxiliary: its equations of motion is algebraic and when the field satisfies its equation of motion (i.e. $e(t)$ is *on shell*), action (2) reduces to the previous one (1).
We consider a generic worldline reparametrization $t \to \tilde{t} = f(t)$ and we let the fields to transform as follows

$$\left\{\begin{array}{ccccc} X^\mu(t) & \longrightarrow & \widetilde{X}^\mu\left(\tilde{t}\,\right) & = & X^\mu(t)\;, \\[2ex] e(t) & \longrightarrow & \tilde{e}\left(\tilde{t}\,\right) & = & e(t)\,\dfrac{dt}{d\tilde{t}}\;, \end{array}\right. \tag{3}$$

then the action (2) is invariant.
The first line in (3) simply tells us that the fields $X^\mu(t)$ are scalars, while one sees from the second line that $e(t)\,dt$ is invariant under worldline reparametrization.
Now, the equation of motion for $e(t)$ reads

$$\frac{\delta S\,[\,X^\mu, e\,]}{\delta e}\;=\;0\qquad\Longrightarrow\qquad\frac{\dot{X}^\mu\dot{X}_\mu}{e^2}+m^2c^2=0\;, \tag{4}$$

and its positive solution is

$$e(t)=\sqrt{-\,\frac{\dot{X}^\mu\dot{X}_\mu}{m^2c^2}}\;. \tag{5}$$

Now it is easy to see that, by substituting the solution of (5) into (2), the action $S\,[\,X^\mu, e\,]$ reduces to $S\,[\,X^\mu\,]$, i.e.

$$S\,[\,X^\mu,\,e\,|_{\text{on shell}}\,]\;=\;S\,[\,X^\mu\,]\;\;. \tag{6}$$

This means that $e(t)$ plays the role of an *auxiliary field.*
The conjugate momentum of the point particle is

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{\dot{X}^\mu}{e} \ . \tag{7}$$

Thus, we observe that the *mass shell condition*

$$p^2 + m^2 c^2 = 0 \tag{8}$$

is equivalent to require that $e(t)$ satisfies its equation of motion.
The constraint in (8) must be imposed on all the physical states also at quantum level.
It is very useful to write the action of the point particle in the following coordinates

$$\begin{cases} X^+ &= \dfrac{X^0 + X^1}{\sqrt{2}} \ , \\ X^- &= \dfrac{X^0 - X^1}{\sqrt{2}} \ , \\ X^i & \qquad\qquad\qquad i = 1, \ldots, D-2 \end{cases} \tag{9}$$

that are known as *light cone coordinates.*
Under a generic diffeomorphism $X^\mu \to \widetilde{X}^\mu$ a D dimensional metric $G_{\mu\nu}(X)$ changes as follows

$$\widetilde{G}_{\mu\nu}(\widetilde{X}) = \frac{\partial X^\alpha}{\partial \widetilde{X}^\mu} \frac{\partial X^\beta}{\partial \widetilde{X}^\nu} \, G_{\alpha\beta}(X) \ . \tag{10}$$

The metric of the flat Minkowskian target space changes accordingly to this prescription under the diffeomorphism in (9) and the light cone metric reads

$$\eta^{LC}_{\mu\nu} = \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \tag{11}$$

where the dots denote the fact that the diagonal must completed with 1.
Given (11) the scalar product between two vectors in light cone basis becomes

$$A^\mu B_\mu = A^\mu B^\nu \eta^{LC}_{\mu\nu} = -A^+ B^- - A^- B^+ + \sum_{i=1}^{D-2} A^i B^i \ . \tag{12}$$

The metric (11) coincides with its inverse and, moreover, the components of a vector A_μ with lower indices are related to the ones of the same vector with higher indices A^μ as follows

$$A_+ = -A^- \ , \qquad A_- = -A^+ \ , \qquad A_i = A^i \ . \tag{13}$$

The action in (2) in light cone coordinates reads

$$S[X^\mu, e] = \frac{1}{2}\int dt \left[-\frac{1}{e}\left(-2\dot{X}^+\dot{X}^- + \sum_{i=1}^{D-2} \dot{X}^{i\,2} \right) + m^2c^2\, e \right] \,, \tag{14}$$

where, as usual, the dot indicates the derivative with respect to t.
The mass shell condition (8) becomes

$$p^2 + m^2c^2 = -2p_+p_- + \sum_{i=1}^{D-2} p_i^2 = 0 \,, \tag{15}$$

and from it one can find

$$p_+ = -\frac{1}{2p_-}\left[\sum_{i=1}^{D-2} p_i^2 + m^2c^2 \right] \,. \tag{16}$$

At this point we fix the reparametrization invariance by setting

$$X^+(t) = t \,. \tag{17}$$

This *gauge fixing* is a special choice of parametrization for the worldline and hereafter reparametrization invariance is no longer manifest. Moreover, because of this gauge fixing, the variable X^+ and its conjugate momentum p_+ are no longer dynamic variables anymore. Indeed, by substituting condition in (17) into the action (14), one sees that the dynamical variables are only X^-, the transverse coordinates X^i and their conjugate momenta

$$p_- = \frac{\partial \mathcal{L}}{\partial \dot{X}^-} = \frac{1}{e} \,, \qquad\qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{X}^i} = -\frac{\dot{X}^i}{e} \,. \tag{18}$$

The Hamiltonian density obtained from (14) is

$$\mathcal{H} = p_-X^- + \sum_{i=1}^{D-2} p_iX^i - \mathcal{L} = -\frac{1}{2p_-}\left[\sum_{i=1}^{D-2} p_i^2 + m^2c^2 \right] \,. \tag{19}$$

Comparing this results with (16), we fully understand the meaning of p_+. Indeed, since X^+ is the time, its conjugate variable p_+ is naturally the Hamiltonian

$$p_+ = \mathcal{H} \,. \tag{20}$$

At this point, we would be ready to quantize the theory, but we will not proceed any further with the point particle because it has been described just to introduce some key concepts that we are going to apply to the study of string theory.

Anyway, the simple case of the point particle has taught to us that the *light cone quantization procedure* requires first to fix the gauge and then to quantize the theory. Its greatest advantage is that all the states introduced in such a way are all the physical states, but the disadvantage consists in loosing the manifest Lorentz covariance.

2 Classical string theory

The basic idea of string theory is that the fundamental object is a one dimensional object, a string instead of a point particle. Then, the different particles that occur in nature are interpreted as the different ways the string can vibrate.
In any case, a string moving in a D dimensional target space describes a two dimensional surface, the *worldsheet*. In the same spirit of the path integral quantization procedure, we will assume that the initial and the final configurations of the string are fixed and known, while the motion of the string through the target space can be given by any two dimensional surface (the path in the point particle language) connecting such configurations.
As for the point particle, where the coordinate $t \in \mathbb{R}$, we will assume that the worldsheet is infinite, i.e. that the range of the coordinate parametrizing the time evolution of the string is $\mathbb{R}$.
Denote by (ξ^0, ξ^1) the worldsheet coordinates; then its immersion in target space is fully described by D *embedding fields*

$$X^\mu(\xi^0, \xi^1) \qquad \mu = 0, 1, \ldots, D-1 \; . \tag{21}$$

They represent the different dimensions of the target space and they are scalar fields under worldsheet reparametrization.
Now, call $G_{\mu\nu}(X)$ the target space metric, the embedding into such a target space induces on the worldsheet a particular metric $G_{ij}(\xi)$ deriving from the requirement that the infinitesimal displacement is invariant

$$d\,s^2 \;=\; G_{\mu\nu}(X)\, dX^\mu dX^\nu \;=\; G_{\mu\nu}(X)\, \frac{\partial X^\mu}{\partial \xi^i}\, \frac{\partial X^\mu}{\partial \xi^j}\, d\xi^i d\xi^j \;=\; G_{ij}(\xi)\, d\xi^i d\xi^j \tag{22}$$

i.e.

$$G_{ij}(\xi) = G_{\mu\nu}(X)\, \frac{\partial X^\mu}{\partial \xi^i}\, \frac{\partial X^\mu}{\partial \xi^j} \; . \tag{23}$$

This is the *induced metric* $G_{ij}(\xi)$ on the worldsheet (with lorentzian signature). Its infinitesimal area element is

$$dA = \sqrt{-\det G_{ij}(\xi)}\; d^2\xi \; . \tag{24}$$

We define an action for the string by mimicing the procedure we used earlier for the point particle. So, we define the action as a quantity proportional to the area of the worldsheet

$$S_{NG}\,[\,X^\mu\,] \;=\; -T \int dA \;=\; -T \int \sqrt{-\det G_{ij}(\xi)}\; d^2\xi \; . \tag{25}$$

This is the *Nambu-Goto action* and its definition is manifestly invariant under worldsheet reparametrization.

The constant of proportionality T is called the *tension* of the string.
At this point, we distinguish between the two worldsheet coordinates by calling $\xi^0 = \tau$ the time coordinate and $\xi^1 = \sigma$ the space one. Their ranges are, respectively,

$$-\infty < \tau < +\infty \qquad \text{and} \qquad 0 \leqslant \sigma \leqslant l \,, \tag{26}$$

where l is the length of the string.
Assuming that the target space is flat, i.e. $G_{\mu\nu}(X) = \eta_{\mu\nu}$, we have

$$G_{ij}(\xi) = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix} \,, \tag{27}$$

where the notation is

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \,, \qquad X' = \frac{\partial X^\mu}{\partial \sigma} \,. \tag{28}$$

Thus, the Nambu-Goto action (25) becomes

$$S_{NG}\left[X^m u\right] \, = \, -T\int d^2\xi \, \sqrt{\left(\dot{X}\cdot X'\right)^2 - \dot{X}^2\, X'^2} \,. \tag{29}$$

The Nambu-Goto action is very natural and easy to understand, but it is very non trivial to quantize, basically because it is non quadratic in the embedding fields X^μ.

The situation is very similar to the case of the point particle, therefore we try to solve this problem as in that case, i.e. by defining a new action quadratic in the fields X^μ but with a new auxiliary non dynamical field γ_{ab} . This new action is known as the *Polyakov action* and it is written as follows

$$S_P\left[X^\mu, \gamma_{ab}\right] \, = \, -\frac{T}{2}\int d\tau d\sigma \sqrt{-\gamma}\,\gamma^{ab}\,\partial_a X^\mu\,\partial_b X^\nu G_{\mu\nu}(X) \, = \, -\frac{T}{2}\int d\tau d\sigma\, \mathcal{L}_P \,. \tag{30}$$

The field γ_{ab} is the worldsheet metric and it is an auxiliary field. Indeed, by using that

$$\delta\sqrt{-\gamma} \, = \, -\frac{1}{2}\sqrt{-\gamma}\,\gamma^{ab}\delta\gamma^{ab} \tag{31}$$

one gets the equations of motion for γ_{ab}

$$\partial_a X^\mu\,\partial_b X^\nu G_{\mu\nu}(X) - \frac{1}{2}\gamma_{ab}\left(\gamma^{cd}\,\partial_c X^\mu\,\partial_d X^\nu G_{\mu\nu}(X)\right) \, = \, 0 \,. \tag{32}$$

Here owe observe that the induced metric

$$\gamma_{ab} \, = \, \partial_a\, X^\mu\,\partial_b X^\nu G_{\mu\nu}(X) \tag{33}$$

solves (32). Moreover, from (32) again, it's easy to see that

$$\det\left[\partial_a\, X^\mu\,\partial_b X^\nu G_{\mu\nu}(X)\right] \, = \, \frac{1}{4}\sqrt{-\gamma}\,\left(\gamma^{cd}\,\partial_c X^\mu\,\partial_d X^\nu G_{\mu\nu}(X)\right)^2 \,. \tag{34}$$

Thus, from this equation it is evident that the Polyakov action (30) recovers the Nambu-Goto action (25) when γ_{ab} satisfies its equation of motion

$$S_P \left[X^\mu, \gamma_{ab}|_{\text{ on shell}} \right] = S_{NG} \left[X^\mu \right] . \tag{35}$$

Concerning the symmetries of the Polyakov action, beside the obvious Lorentz and two dimensional diffeomorphisms invariances, there is a further one, known as *Weyl symmetry.* It does not affect the coordinates, but it changes the worldsheet metric by an arbitrary positive factor

$$\gamma_{ab}(\tau, \sigma) \longrightarrow \Omega(\tau, \sigma)^2 \, \gamma_{ab}(\tau, \sigma) . \tag{36}$$

If the requirement is just two-dimensional diffeomorphisms invariance, then other terms are allowed. In particular, we can add an *Einstein-Hilbert* term

$$\lambda \int \sqrt{-\gamma} \, R \, d^2\xi , \tag{37}$$

where R is the scalar curvature of the worldsheet metric, as well as a cosmological term

$$\Lambda \int \sqrt{-\gamma} \, d^2\xi . \tag{38}$$

This term is proportional to the worldsheet area and Λ is called *cosmological constant.* Note that, while the Einstein-Hilbert term is also Weyl invariant, the cosmological term is not.

In two dimensions, the Einstein-Hilbert term becomes a *topological* quantity, i.e. it is independent of the metric of the worldsheet. Indeed, a fundamental result obtained within the study of the Riemann surfaces is the *Gauss Bonnet theorem*, which states that

$$\int \sqrt{-\gamma} \, R \, d^2\xi = 4\pi \, \chi \tag{39}$$

where χ is the *Euler characteristic* of the Riemann surface. For a Riemann surface of *genus* g and b boundaries, χ is given by

$$\chi = 2 - 2\,g - b . \tag{40}$$

For the time being, we neglect these terms, but we will discuss them within the study of the bosonic string on nontrivial backgrounds.

Given the Polyakov action, we can consider its *energy momentum tensor* T_{ab}.

When the action of the theory depends on a Lorentzian metric γ_{ab}, the general definition of T_{ab} is

$$T_{ab} = - \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ab}} . \tag{41}$$

For our Polyakov action (30), the energy momentum tensor is given by

$$T_{ab} = \frac{1}{T} \left(\partial_a X^\mu \, \partial_b X^\nu G_{\mu\nu}(X) - \frac{1}{2} \gamma_{ab} \left(\gamma^{cd} \, \partial_c X^\mu \, \partial_d X^\nu G_{\mu\nu}(X) \right) \right) . \tag{42}$$

Note that, from the definition of the equations of motion for γ_{ab}, we have:

$$\gamma_{ab}\big|_{\text{on shell}} \qquad \Longleftrightarrow \qquad T_{ab} = 0 . \tag{43}$$

This tensor is traceless and conserved, i.e.

$$\gamma^{ab} \, T_{ab} = 0 \qquad \qquad \text{tracelessness,} \tag{44}$$

$$\nabla^a \, T_{ab} = 0 \qquad \qquad \text{conservation.} \tag{45}$$

Note that tracelessness holds also off shell.

Now we consider the equations of motion for the embedding fields $X^\mu(\xi)$, whose discussion needs more attention.
By setting to 0 the variation of the Polyakov action with respect to X^μ, one gets:

$$\int d\tau \, d\sigma \, \delta X^\mu \, \partial_a \left(\sqrt{-\gamma} \, \gamma^{ab} \, \partial_b X^\nu \, G_{\mu\nu} \right) - \int d\tau \, d\sigma \, \partial_a \left(\delta X^\mu \sqrt{-\gamma} \, \gamma^{ab} \, \partial_b X^\nu \, G_{\mu\nu} \right) = 0 . \tag{46}$$

The first term is a bulk term and the second one gives rise to boundary terms.
Imposing the vanishing of the bulk term, we get the equations of motion for the embedding fields $X^\mu(\xi)$

$$\partial_a \left(\sqrt{-\gamma} \, \gamma^{ab} \, \partial_b X^\nu \, G_{\mu\nu}(X) \right) = 0 . \tag{47}$$

In particular, when the target space is Minkowsky flat, i.e. $G_{\mu\nu}(X) = \eta_{\mu\nu}$, we have

$$\partial_a \left(\sqrt{-\gamma} \, \gamma^{ab} \, \partial_b X^\nu \right) \eta_{\mu\nu} = \sqrt{-\gamma} \, \Delta_{LB} \, X_\mu = 0 \tag{48}$$

where Δ_{LB} is the *Laplace Beltrami operator* associated with the lorentzian metric γ_{ab}.
The second term in (46) generates some boundary terms, whose vanishing depends on the boundary conditions imposed on the fields. As we will see here, such boundary conditions determinate the different kinds of strings we can deal with.
The boundary term computed at the two end of the string is

$$\int_{\tau_1}^{\tau_2} d\tau \;\; \delta X^\mu \sqrt{-\gamma} \, \gamma^{\sigma b} \, \partial_b X^\nu \, G_{\mu\nu} \Big|_{\sigma=0}^{\sigma=l} = 0 . \tag{49}$$

By requiring that it vanishes, we fix the possible boundary conditions $X^\mu(\tau, 0)$ and $X^\mu(\tau, l)$ we can impose on the embedding fields at the spatial extrema.
In particular, the vanishing of (49) holds in the following cases

- Open strings

  - fixed spatial extrema: $\delta X^\mu\big|_{\sigma=0} \;=\; \delta X^\mu\big|_{\sigma=0} \;=\; 0$ *Dirichlet b.c.*
  - free spatial extrema: $\gamma^{\sigma b}\,\partial_b\,X^\mu\Big|_{\sigma=0} \;=\; \gamma^{\sigma b}\,\partial_b\,X^\mu\Big|_{\sigma=l} \;=\; 0$ *Neumann b.c.*
  - mixed boundary conditions:
    Dirichlet at one spatial extremum and Neumann at the other or viceversa.

- Closed strings

  - $X^\mu(\tau,\sigma+l\,) \;=\; X^\mu(\tau,\sigma)\;.$

In the previous discussions, we have not fixed any gauge.
Now, to get the fundamental results in a simpler way, we fix a convenient gauge by using the three local invariances of the action, i.e. the Weyl invariance (36) and the two-dimensional worldsheet diffeomorphisms invariance.
First, we use two of our three gauge invariances to put the worldsheet metric in the form

$$\gamma_{\sigma\tau} \;=\; 0 \qquad\qquad \gamma \;=\; \det\gamma_{ab} \;=\; -1\;. \tag{50}$$

This implies that $\gamma_{\tau\tau} = 1/\gamma_{\sigma\sigma}$.
It is always possible to reach the gauge (50) by a worldsheet coordinates change of the form

$$\left\{\begin{array}{lcl} \tilde\sigma & \longrightarrow & \sigma \;=\; \sigma\,(\,\tilde\sigma,\tilde\tau\,)\;, \\ \tilde\tau & \longrightarrow & \tau \;=\; \tilde\tau\;. \end{array}\right. \tag{51}$$

Indeed, if one starts with a generic metric $\tilde\gamma_{ab}$, then off diagonal component of the metric in the new coordinates (σ,τ) defined in (51) is

$$\gamma_{\sigma\tau} \;=\; \frac{\partial\tilde\sigma}{\partial\tau}\,\frac{\partial\tilde\sigma}{\partial\sigma}\,\tilde\gamma_{\tilde\sigma\tilde\sigma} + \frac{\partial\tilde\sigma}{\partial\sigma}\,\tilde\gamma_{\tilde\sigma\tilde\tau}\;. \tag{52}$$

Thus

$$\gamma_{\sigma\tau} \;=\; 0 \qquad\Longrightarrow\qquad \tilde\sigma \;=\; -\int d\tau\,\frac{\tilde\gamma_{\tilde\sigma\tilde\sigma}}{\tilde\gamma_{\tilde\sigma\tilde\tau}}\;. \tag{53}$$

Then, through a Weyl transformation, we can always send the metric with $\gamma_{\sigma\tau} \;=\; 0$ into a new one with determinant equal to -1 as follows

$$\gamma_{ab} \;\longrightarrow\; \Omega^2\,\gamma_{ab} \qquad \text{with} \qquad \Omega^2 \;=\; -\,\frac{1}{\gamma_{\sigma\sigma}\,\gamma_{\tau\tau}}\;. \tag{54}$$

Notice that we have given only two constraints, therefore we are left with one more degree of freedom.

At this point, we choose the light cone coordinates (X^+, X^-, X^i) given in (9) for the target space because in such system is much more easier to impose the last constraint that fixes the gauge completely.
Indeed, as for the point particle case, our last gauge fixing condition is

$$X^+(\sigma,\tau) \,=\, \tau \;. \tag{55}$$

Summing up, we have used all our gauge freedom to fix the following three conditions

$$\begin{cases} X^+ &= \;\; \tau \;, \\ \gamma_{\sigma\tau} &= \;\; 0 \;, \\ \gamma &= \;\; -1 \;, \end{cases} \tag{56}$$

which we will work with during this section.

The Polyakov action in the Minkowski target space, parametrized by light cone coordinates, is

$$S_P\;[\,X^\mu, \gamma_{ab}\,] \,=\, -\frac{T}{2}\int d\tau d\sigma \sqrt{-\gamma}\,\left(-2\,\gamma^{ab}\,\partial_a X^+\,\partial_b X^- + \sum_{i=1}^{D-2} \gamma^{ab}\,\partial_a X^i\,\partial_b X^i \right) \;. \tag{57}$$

In the gauge given by (56), it becomes

$$\begin{aligned} S_P\;[\,X^\mu, \gamma_{ab}\,]\,|_{\,\text{gauge fixed}} \;\; &= \\ -\frac{T}{2}\int d\tau d\sigma &\left(2\,\gamma_{\sigma\sigma}\,\partial_\tau X^- + \sum_{i=1}^{D-2} \left(-\gamma_{\sigma\sigma}\,\partial_\tau X^i\,\partial_\tau X^i + \frac{1}{\gamma_{\sigma\sigma}}\,\partial_\sigma X^i\,\partial_\sigma X^i \right)\right) \;. \end{aligned} \tag{58}$$

In this gauge one finds as degrees of freedom X^-, the transverse coordinates X^i and their conjugate momenta, that are, respectively

$$p_- \,=\, -\,p^+ \,=\, -\,T\,\gamma_{\sigma\sigma} \;, \qquad \Pi_i \,=\, \Pi^i \,=\, T\,\gamma_{\sigma\sigma}\,\dot X^i \;, \tag{59}$$

where the dot represents the derivative with respect to τ.
From (58), the classical Hamiltonian density is

$$\mathcal{H} \,=\, p_-\,\dot X^- + \sum_{i=1}^{D-2} \Pi_i\,\dot X^i - \mathcal{L} \,=\, \frac{T}{2}\sum_{i=1}^{D-2}\left(\gamma_{\sigma\sigma}\,\left(\partial_\tau X^i\right)^2 + \frac{1}{\gamma_{\sigma\sigma}}\,\left(\partial_\sigma X^i\right)^2 \right) \;. \tag{60}$$

The equation of motion for $\gamma_{\sigma\sigma}$ gives

$$\frac{\delta\,S_P}{\delta\,\gamma_{\sigma\sigma}} \,=\, 0 \qquad \Longrightarrow \qquad 2\,\partial_\tau\,X^- \,=\, \sum_{i=1}^{D-2}\left(\partial_\tau X^i\,\partial_\tau X^i + \frac{1}{\gamma_{\sigma\sigma}^2}\,\partial_\sigma X^i\,\partial_\sigma X^i \right) \;, \tag{61}$$

that can be written in terms of the momenta (59) as

$$2\,\partial_\tau X^- \;=\; \frac{1}{\gamma_{\sigma\sigma}^2}\sum_{i=1}^{D-2}\left(\left(\frac{\Pi^i}{T}\right)^2 + \left(\partial_\sigma X^i\right)^2\right) \;. \tag{62}$$

Thus, in this gauge, the requirement that γ_{ab} is on shell means that (62) holds.
Now we consider the equations of motion for the embedding fields.
From (58), the equation of motion for X^- reads

$$\partial_\tau\,\gamma_{\sigma\sigma} \;=\; 0\;, \tag{63}$$

i.e. $\gamma_{\sigma\sigma} \;=\; \gamma_{\sigma\sigma}(\sigma)$.
In our gauge, the equation of motion for the transverse coordinate X^i can be obtained from the general expression (47) by first inserting $G_{\mu\nu}(X) \;=\; \eta^{LC}_{\mu\nu}$ and the gauge fixing (56) and then taking the ith component. The result is

$$\partial_\sigma\left(\gamma_{\sigma\sigma}^{-1}\,\partial_\sigma X^i\right) - \partial_\tau\left(\gamma_{\sigma\sigma}\,\partial_\tau X^i\right) \;=\; 0\;. \tag{64}$$

Now, we can still perform a reparametrization of $\sigma = \sigma\,(\,\tilde\sigma\,)$ alone to render $\gamma_{\sigma\sigma}$ constant at any given time $(\,\tau\;=\;0$ for instance$\,)$. Such a reparametrization does not change the gauge conditions (56). Then, the equation of motion for X^- (63) implies that $\gamma_{\sigma\sigma}$ is constant for any time:

$$\gamma_{\sigma\sigma} \;=\; \frac{p^+}{l\,T} \;\equiv\; \frac{1}{C} \;=\; \text{const}\;. \tag{65}$$

So, (64) becomes

$$\left(C^2\,\partial_\sigma^2 - \partial_\tau^2\right)X^i \;=\; 0\;. \tag{66}$$

We could reparametrize the time in such a way that $C = 1$ as in [GSW], but we choose to keep C as [P] does, for the time being.
Since we want L^2 functions and the set

$$\left\{\exp\left(-\,i\,\frac{2\pi n\;(\sigma + C\tau)}{l}\right)\;,\;\exp\left(-\,i\,\frac{2\pi n\;(\sigma - C\tau)}{l}\right)\;,\;n\in\mathbb{Z}\right\} \tag{67}$$

is a complete basis of L^2 that satisfy (66), we can expand the general solution in the basis (67).
To give a complete solution of our classical theory, we must give only the transverse coordinates X^i and

$$x^- \;=\; \frac{1}{l}\int_0^l d\sigma\,X^-(\sigma)\;. \tag{68}$$

Indeed, all the other components are given by the gauge fixing $(\,X^+ = \tau)$ and the equation of motion for γ_{ab}, (62).

The general solution of (66) depends on the boundary conditions one imposes on the fields X^i at the spatial extrema.
First we note that *Neumann boundary conditions*,

$$\gamma^{\sigma b}\,\partial_b\,X^\mu\big|_{\sigma=0} \;=\; \gamma^{\sigma b}\,\partial_b\,X^\mu\big|_{\sigma=l} \;=\; 0\;, \tag{69}$$

in our gauge choice reduce to

$$\partial_\sigma\,X^\mu\big|_{\sigma=0} \;=\; \partial_\sigma\,X^\mu\big|_{\sigma=l} \;=\; 0\;. \tag{70}$$

Therefore the solution for the open string with free spatial extrema in the transverse coordinates (i.e. Neumann b.c. (70) with $\mu = i$) is

$$X^i(\tau,\sigma) \;=\; x^i(0) + \frac{p^i(0)}{p^+}\,\tau + i\;(2\alpha')^{1/2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{\alpha^i_n}{n}\,\exp\left(-\,i\,\frac{\pi n C\tau}{l}\right)\cos\frac{\pi n\sigma}{l}\;. \tag{71}$$

As for the Dirichlet boundary conditions, since they need a more careful discussion we will study them better later.
For the closed string boundary conditions,

$$X^i(\tau,\sigma+l) \;=\; X^i(\tau,\sigma)\;, \tag{72}$$

the solution is instead

$$\begin{aligned} X^i(\tau,\sigma) \;&=\; x^i(0) + \frac{p^i(0)}{p^+}\,\tau \qquad\qquad (73)\\ &+ i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left\{\frac{\alpha^i_n}{n}\,\exp\left(-\,i\,\frac{2\pi n\,(\sigma + C\tau)}{l}\right) + \frac{\tilde{\alpha}^i_n}{n}\,\exp\left(-\,i\,\frac{2\pi n\,(\sigma - C\tau)}{l}\right)\right\}\;. \end{aligned}$$

Notice that we have slightly changed the notation by introducing the parameter α', that is related to the string tension T as follows

$$T \;=\; \frac{1}{2\pi\,\alpha'}\;. \tag{74}$$

At this level, the main difference between the open string and the closed string is that while in the open case only an infinite set of oscillators appears, in the closed one there are two independent sets of oscillating modes, α^i_n and $\tilde{\alpha}^i_n$ (indeed, if we set $\tilde{\alpha}^i_n = \alpha^i_n$ in the closed string solution (73) we find the open string solution (71) with Neumann b.c.). They corresponds respectively to the left moving and the right moving waves along the string.

In the mode decompositions we have also introduced the *center of mass variables*, defined as

$$x^i(\tau) \;=\; \frac{1}{l}\int_0^l d\sigma\, X^i(\tau,\sigma)\;, \tag{75}$$

$$p^i(\tau) \;=\; \frac{1}{l}\int_0^l d\sigma\, \Pi^i(\tau,\sigma)\;, \tag{76}$$

i.e. the mean values of the position and the momenta at a fixed time τ.
Moreover the reality of the embedding fields implies

$$X^i(\tau,\sigma)\in\mathbb{R} \qquad \Longrightarrow \qquad \alpha^i_{-n} \,=\, \left(\alpha^i_n\right)^\dagger \quad \text{and} \quad \tilde{\alpha}^i_{-n} \,=\, \left(\tilde{\alpha}^i_n\right)^\dagger\;, \tag{77}$$

therefore the independent modes have only $n > 0$.
To close this section, we write the classical Hamiltonian density (60) by using the expression of $\gamma_{\sigma\sigma}$ given in (65)

$$\mathcal{H} \,=\, \frac{p^+}{2\,l}\sum_{i=1}^{D-2}\Big(\left(\partial_\tau X^i\right)^2 + C^2\,\left(\partial_\sigma X^i\right)^2\Big)\;. \tag{78}$$

Moreover, we recall that the Hamiltonian is

$$H \,=\, \int_0^l \mathcal{H}\, d\sigma\;. \tag{79}$$

This formulas will be applied in the next chapter.

3 Open string with Neumann b.c. : light cone quantization

In this section, we will quantize the open string theory with Neumann boundary conditions in the light cone coordinates and with the gauge fixed in (56), by following the *canonical quantization procedure* for the Hamiltonian theory.
First, we recall the solutions of the equations of motion for the transverse coordinates with Neumann b.c.

$$X^i(\tau,\sigma) \;=\; x^i(0) + \frac{p^i(0)}{p^+}\,\tau + i\,\left(2\alpha'\right)^{1/2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\alpha^i_n}{n}\, \exp\left(-\,i\,\frac{\pi n C\tau}{l}\right) \cos\frac{\pi n\sigma}{l}\;. \qquad (80)$$

As for the X^- coordinate, we recall that its equation of motion gives the condition $\partial_\tau\,\gamma_{\sigma\sigma} = 0$, but, since it is a L^2 function as well, we can expand it as follows

$$X^-(\tau,\sigma) \;=\; x^-(0) + \frac{p^-(0)}{p^+}\,\tau + i\,\left(2\alpha'\right)^{1/2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\alpha^-_n}{n}\, \exp\left(-\,i\,\frac{\pi n C\tau}{l}\right) \cos\frac{\pi n\sigma}{l}\;. \qquad (81)$$

This expression, apart $x^-(0)$, is completely determined from the constraint (62), that is

$$2\,\partial_\tau X^- \;=\; C^2 \sum_{i=1}^{D-2} \left(\left(\frac{\Pi^i}{T}\right)^2 + \left(\partial_\sigma X^i\right)^2 \right)\;. \qquad (82)$$

coming from the equation of motion for $\gamma_{\sigma\sigma}$.
Since

$$\int_0^l \cos\frac{\pi n\sigma}{l}\,\cos\frac{\pi n\sigma}{l}\,d\sigma \;=\; \int_0^l \sin\frac{\pi n\sigma}{l}\,\sin\frac{\pi n\sigma}{l}\,d\sigma \;=\; \frac{l}{2}\,\delta_{m,n}\;, \qquad (83)$$

then the Hamiltonian

$$H \;=\; \int_0^l \mathcal{H}\,d\sigma \;=\; \int_0^l \frac{p^+}{2\,l} \sum_{i=1}^{D-2} \left(\left(\partial_\tau X^i\right)^2 + C^2\,\left(\partial_\sigma X^i\right)^2 \right) d\sigma \qquad (84)$$

becomes

$$H \;=\; \frac{p^i p^i}{2p^+} + \frac{1}{4\alpha' p^+} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \alpha^i_n\,\alpha^i_{-n}\;. \qquad (85)$$

In the classical theory the modes commutes and we can write the Hamiltonian as follows

$$H \;=\; \frac{p^i p^i}{2p^+} + \frac{1}{2\alpha' p^+} \sum_{n=1}^{\infty} \alpha^i_{-n}\,\alpha^i_n\;. \qquad (86)$$

Until now, we have studied string theory only at classical level.

Our classical theory is written in light cone coordinates and in the gauged fixed by (56). To quantize this theory in the canonical way we must promote the coordinates and their conjugate momenta to operators acting on some Hilbert space of states.
Thus, in the quantum theory $X^\mu(\tau,\sigma)$ and $\Pi^\mu(\tau,\sigma)$ are operators which satisfy the *canonical commutation relations at equal time*, i.e.

$$[\,X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\,] \;=\; i\,\delta(\sigma-\sigma')\,\eta^{\mu\nu}\ , \tag{87}$$

$$[\,X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')\,] \;=\; [\,\Pi^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\,] \;=\; 0\ , \tag{88}$$

where $\eta^{\mu\nu}$ is the inverse of the metric $\eta^{LC}_{\mu\nu}$.
Since X^μ are operators, also their Fourier coefficients are operators, therefore we can equivalently express (87) and (88) as commutation relations between modes. Indeed, by inserting X^μ given in (80) and (81) into the commutators (87) and (88), we find that for the open string with Neumann b.c. the canonical commutation relations are equivalent to the following equal time commutation rules

$$[\,x^\mu,p^\nu\,] \;=\; i\,\eta^{\mu\nu}\ , \tag{89}$$

$$[\,x^\mu,\alpha^\nu_n\,] \;=\; [\,p^\mu,\alpha^\nu_n\,] \;=\; 0\ , \tag{90}$$

$$[\,\alpha^\mu_n,\alpha^\nu_m\,] \;=\; n\,\delta_{m+n,0}\,\eta^{\mu\nu}\ , \tag{91}$$

where $x^\mu = x^\mu(0)$ and $p^\mu = p^\mu(0)$ are center of mass operators [K].
To define the Hilbert space of the states, we introduce the *vacuum state* $|\,0,k\rangle$ with $k=(k^+,k^i)$, $i=1,\ldots,D-1$ as the state satisfying

$$p^+\,|\,0,k\,\rangle \;=\; k^+\,|\,0,k\,\rangle\ , \tag{92}$$

$$p^i\,|\,0,k\,\rangle \;=\; k^i\,|\,0,k\,\rangle\ , \tag{93}$$

$$\alpha^i_n\,|\,0,k\rangle \;=\; 0 \qquad\qquad n>0\ . \tag{94}$$

The last equations lead us to define the operators α^i_n with $n>0$ as the *destruction operators.*
All the states of the Hilbert space are given by all the possible way to excite the vacuum state by *creation operators* $\alpha^i_{-n} = (\,\alpha^i_n\,)^\dagger$ with $n>0$

$$\prod_{i=1}^{D-2}\prod_{n>0}\alpha^i_{-n}\;|\,0,k\,\rangle \tag{95}$$

Note that in the light cone quantization all the states one can introduce are all physical states.

A careful discussion is needed for the Hamiltonian operator. The classical Hamiltonian is given by (86). In the quantum theory, α_n^i are operators and they do not commute. Since at fixed n and i they satisfy the same commutation rules of the harmonic oscillator, when we order the operators as in (86), we have a constant for each i and n. In the sum over n, this gives rise to an infinite term.

Because of this phenomenon, we adopt as the *Hamiltonian operator* the following expression

$$H = \frac{p^i p^i}{2p^+} + \frac{1}{2\alpha' p^+} \left(\sum_{n=1}^{\infty} \alpha_{-n}^i \, \alpha_n^i + A \right) , \tag{96}$$

where A is a constant that takes into account of all the constants coming from the normal ordering of the modes. For the beginning, A will be a generic constant. Its value will be fixed as a necessary consequence of the Lorentz invariance of the quantum theory.

Notice that, with respect to the point particle case, where only the time evolution along the worldline was present, here we have also the coordinate σ, therefore the relevant quantity is the Hamiltonian $H(\tau)$, and not the Hamiltonian density $\mathcal{H}(\tau, \sigma)$.

In general, for the wave function of a state, we have that

$$\psi(\tau + \Delta\tau, X^-, X^i) = \psi(\tau, X^-, X^i) + \Delta\tau \, \frac{\partial \psi}{\partial \tau} = \psi(\tau, X^-, X^i) - i \, \Delta\tau \, H \, \psi \, . \tag{97}$$

In our case, since we have fixed $X^+ = \tau$, this becomes

$$\begin{aligned} \psi(\tau + \Delta\,\tau, X^-, X^i) &= \psi(X^+ + \Delta X^+, X^-, X^i) = \psi(X^+, X^-, X^i) + \Delta X^+ \, \frac{\partial \psi}{\partial X^+} \\ &= \psi(X^+, X^-, X^i) + i \, \Delta X^+ \, p_+ \, \psi \\ &= \psi(\tau, X^-, X^i) + i \, \Delta\tau \, p_+ \, \psi \, . \end{aligned} \tag{98}$$

Therefore

$$H = - p_+ = p^- \, . \tag{99}$$

From (100), we have that

$$2p^+ p^- - p^i p^i = \frac{1}{2\alpha'} \sum_{n=1}^{\infty} \alpha_{-n}^i \, \alpha_n^i + A \, . \tag{100}$$

We observe that in the r.h.s of this equation we have $-p^2$ written in the light cone coordinates. Then, the *mass shell conditions* tells that $p^2 + M^2 = 0$ (hereafter we will adopt $c = 1$ units). Therefore, we are led to define the *squared mass operator* as

$$M^2 = \frac{1}{\alpha'} \left(\sum_{n>0} \alpha_{-n}^i \, \alpha_n^i + A \right) \, . \tag{101}$$

All the states (95) are eigenstates of the mass operator M^2.
For instance, the vacuum states is an eigenstate with eigenvalue A/α', i.e.

$$M^2\, |\, 0,k\rangle \;=\; \frac{A}{\alpha'}\, |\, 0,k\rangle \;\;. \qquad (102)$$

The first level excited states are the $D-2$ states given by $\alpha^i_{-1}\, |\, 0,k\rangle$ and they have mass

$$M^2\alpha^i_{-1}\, |\, 0,k\rangle \;=\; \frac{1+A}{\alpha'}\, \alpha^i_{-1}\, |\, 0,k\rangle \qquad i=1,\ldots,D-2\,. \qquad (103)$$

We can continue to construct other one level excited states with increasing masses

$$M^2\alpha^i_{-r}\, |\, 0,k\rangle \;=\; \frac{r+A}{\alpha'}\, \alpha^i_{-r}\, |\, 0,k\rangle \qquad r \geqslant 2 \qquad i=1,\ldots,D-2\,. \qquad (104)$$

Now, if we focus on the first level excited states (103), we observe that, since they are $D-2$, they belong to to a representation of $SO(D-2)$. But $SO(D-2)$ is a representation of the Lorentz group only if the particle associated is *massless*. Therefore, the first level excited states must describe a massless particle and, for the open strings with Neumann b.c., this occurs only if

$$A \;=\; -1\;. \qquad (105)$$

Notice that we have not demonstrated the Lorentz invariance of our gauge fixed theory. We have just assumed it, and this assumption has given $A=-1$ as a necessary consequence .
The value (105) for A implies that the vacuum $|\, 0,k\rangle$ is *tachyon*, i.e. a particle with *negative mass*

$$M^2\, |\, 0,k\,\rangle \;=\; -\frac{1}{\alpha'}\, |\, 0,k\,\rangle \;\;. \qquad (106)$$

The string theory was born as a theory of strong interactions and in that context the vacuum state was identified with the pion. Clearly, having a tachyonic pion was quite embarassing and this was one of the reasons, together with the occurring of the critical dimension, that induced to abandon string theory as a theory of the strong interactions. Anyway, the problem of the tachyonic vacuum can be solved within the supersymmetric extension of the theory through the GSO projection, as we will see later.
As has been often remarked, having fixed the gauge, we have lost the manifest Lorentz invariance of our theory. We want that, though not manifestly, our theory *is* Lorentz invariant. Therefore, we assume it and we study what this invariance tell us.
The generator of the Lorentz transformations given by the Noether theorem is

$$J^{\mu\nu} \;=\; x^\mu p^\nu - p^\mu x^\nu - i\sum_{n>0}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\mu_n\alpha^\nu_{-n}\right)\;. \qquad (107)$$

Its component J^{i-} in the light cone coordinates is

$$J^{i-} \;=\; x^i p^- - p^i x^- - i \sum_{n>0} \frac{1}{n} \left(\alpha^i_{-n} \alpha^-_n - \alpha^i_n \alpha^-_{-n} \right) \; . \tag{108}$$

Since the operators $\alpha^-_{\pm n}$ can be obtained from the constraint (62), they are quadratic expressions of $\alpha^i_{\pm n}$. Therefore J^{i-} is a cubic expression of the modes $\alpha^i_{\pm n}$.
The Lorentz algebra reads

$$\left[J^{\mu\nu}, J^{\rho\lambda} \right] \;=\; -i\,\eta^{\mu\rho} J^{\mu\lambda} + i\,\eta^{\mu\rho} J^{\nu\lambda} + i\,\eta^{\nu\lambda} J^{\mu\rho} - i\,\eta^{\mu\lambda} J^{\nu\rho} \; , \tag{109}$$

and, in particular, we have that

$$\left[J^{i-}, J^{j-} \right] \;=\; 0 \; . \tag{110}$$

Therefore, if (109) holds, it is necessary that

$$\langle\, 0, k \,|\; \alpha^i_2 \; \left[J^{i-}, J^{j-} \right] \; \alpha^i_{-2} \;|\, 0, k \,\rangle \;=\; 0 \; . \tag{111}$$

Instead, if we computed the r.h.s. by using the canonical commutation relations, we find that

$$\langle\, 0, k \,|\; \alpha^i_2 \; \left[J^{i-}, J^{j-} \right] \; \alpha^i_{-2} \;|\, 0, k \,\rangle \;\propto\; D \,-\, 26 \; . \tag{112}$$

Therefore, if we want a Lorentz invariant quantum theory, then we are forced to work with

$$D \;=\; 26 \; . \tag{113}$$

This is the *critical spacetime dimension of the bosonic string theory* and it is another necessary condition deriving from the Lorentz invariance of the quantum theory.

REGGE TRAJECTORIES

DIRICHLET BOUNDARY CONDITIONS

UNORIENTED OPEN STRINGS

CLOSED STRINGS

4 Worldsheet 2D supersymmetry

To describe fermionic particles within the string theory, we introduce other D fields $\psi^\mu(\tau,\sigma)$ beside the embedding scalar fields $X^\mu(\tau,\sigma)$. These fields $\psi^\mu(\tau,\sigma)$ are vectors under the Lorentz transformations of the target space, but they are also *two dimensional Majorana spinors*.

Let us fix some definitions.
In the Majorana basis, the two dimensional Dirac matrices read

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \tag{114}$$

In this basis, they satisfy the algebra

$$\{ \rho^a, \rho^b \} = -2\, \eta^{ab} , \tag{115}$$

where η^{ab} is a matrix of matrices given by

$$\eta^{ab} = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} , \tag{116}$$

and $\mathbb{I}$ is the 2×2 identity matrix.
The Dirac conjugate of the field ψ is defined $\bar{\psi} \equiv \psi^\dagger \rho^0$, as usual.
The Majorana condition, that defines the Majorana spinors, is

$$\psi^* = \psi . \tag{117}$$

This means that we can write our two dimensional Majorana spinors as follows

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} , \tag{118}$$

where the components $\psi_\pm$ are *real* functions of (τ,σ).
Therefore, the Dirac conjugate of a Majorana spinor ψ is simply $\bar{\psi} = \psi^t \rho^0$.
Now we introduce a further Dirac matrix

$$\rho^3 = \rho^0 \rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{119}$$

that plays the same role of γ^5 in four dimensions.
Indeed, through ρ^3 we can define the projection operators

$$\frac{1+\rho^3}{2} \qquad \text{and} \qquad \frac{1-\rho^3}{2} . \tag{120}$$

They allow us to extract the different components of the Majorana spinors as follows

$$\frac{1+\rho^3}{2}\,\psi = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} \qquad \text{and} \qquad \frac{1-\rho^3}{2}\,\psi = \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} . \tag{121}$$

Since

$$\frac{1+\rho^3}{2}\begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} \qquad \text{and} \qquad \frac{1-\rho^3}{2}\begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} , \tag{122}$$

these spinors are Weyl spinors.

Now we want to write an action depending on X^μ, ψ^μ and eventually some other auxiliary, nondynamical fields, that permit us to get the analog of the Nambu-Goto action when they are on shell.
In the bosonic string theory, the auxiliary field is the two dimensional metric γ_{ab}. The kinetic term of the field X^0 is negative, i.e. X^0 is a *ghost*. This implies that, in absence of the constraints following from the equation of motion for γ_{ab}, negative norm states occur or, alternatively, an unbounded energy spectrum from below occur. Instead, when γ_{ab} is on shell, these problems are avoided and we recover the Nambu-Goto action.
In the fermionic string theory, if we add just the two dimensional worldsheet metric $h_{\alpha\beta}$ (note the different notation for the worldsheet curved indices with respect to the bosonic case), we do not get the true analog of the Nambu-Goto action when we eliminate $h_{\alpha\beta}$ through its equation of motion. We must add also another nondynamical field

$$\chi_\alpha = \begin{pmatrix} \chi_\alpha^+ \\ \chi_\alpha^- \end{pmatrix} . \tag{123}$$

This field is the *gravitino*: it is a vector of the D dimensional Lorentz group and a two dimensional Majorana spinor.
The worldsheet metric together with the gravitino are enough to eliminate all the oscillators coming from the ghost fields, that are X^0 and ψ^0 for the fermionic string.

Before to address the writing of the action, we fix that hereafter we will adopt the [GSW] notation, i.e.

$$\alpha' = \frac{1}{2} = l_s^2 , \qquad l = \pi , \qquad \tau = \frac{2\pi\alpha' p^+}{l}\,\tau^{GSW} = \frac{\tau^{GSW}}{C} . \tag{124}$$

Moreover, we will drop also the label and for us it will be implicit that $\tau = \tau^{GSW}$.

With the notation established above, now we introduce the following action for the fermionic string :

$$S_0\left[X^\mu,\psi^\mu,h_{\alpha\beta}\right] = -\frac{1}{2\pi}\int d\tau d\sigma\,\sqrt{-h}\,\left(h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\eta_{\mu\nu} - i\,\bar\psi^\mu\rho^\alpha\nabla_\alpha\psi^\nu\eta_{\mu\nu}\right)\ . \quad (125)$$

The changing of notation for the worldsheet curved indices is due to the fact that now it is convenient to express the worldsheet curved metric in terms of the two dimensional flat minkowskian metric η_{ab}

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\ . \quad (126)$$

This can be done through the *zweibein* e^a_α as follows

$$h_{\alpha\beta} = e^a_\alpha\, e^b_\beta\,\eta_{ab} \qquad \Longrightarrow \qquad h^{\alpha\beta} = e^\alpha_a\, e^\beta_b\,\eta^{ab} \quad (127)$$

The latin indices (a,b) are two dimensional flat indices and the curved nature of $h_{\alpha\beta}$ is completely encoded into the one forms e^a_α. With these definitions, we can also introduce the one forms ω^{ab}_α

$$\omega^{ab}_\alpha = \omega^{ab}_\alpha\, d\sigma^\alpha \quad (128)$$

defined as the solutions of

$$de^{\,a} + \omega^{ab}\wedge e^b = 0 \qquad \text{i.e.} \qquad D\,e^a = 0\ . \quad (129)$$

Such one forms ω^{ab}_α are known as *spin connection.*
In terms of the zweibein and the spin connections, the Dirac matrices with curved indices are

$$\rho^\alpha = e^\alpha_a\,\rho^a\ , \quad (130)$$

while the covariant derivative reads

$$\nabla_\alpha\psi = \partial_\alpha\psi + \omega^{ab}_\alpha\,\frac{\rho^a\rho^b - \rho^b\rho^a}{4}\,\psi\ . \quad (131)$$

The term

$$\frac{\rho^a\rho^b - \rho^b\rho^a}{4} \quad (132)$$

is the generator of the Lorentz rotations in the spinorial notation.
The action (125) is manifestly invariant under two dimensional diffeomorphisms, but we require also the invariance under two dimensional supersymmetric transformations.
The *two dimensional worldsheet supersymmetry transformations* are

$$\begin{cases} \delta\,X^\mu = \bar\epsilon\,\psi^\mu\ , \\ \delta\,\psi^\mu = -i\,\rho^\alpha\,\epsilon\,\left(\partial_\alpha X^\mu - \bar\psi^\mu\chi_\alpha\right)\ , \\ \delta\,e^a_\alpha = -2i\,\bar\epsilon\,\rho^a\chi_\alpha\ , \\ \delta\,\chi_\alpha = \nabla_\alpha\,\epsilon\ . \end{cases} \quad (133)$$

The constant ϵ does not depend on the world sheet coordinates (global supersymmetry) is the infinitesimal parameter generating the worldsheet supersymmetry, likewise $\xi^\alpha(\tau, \sigma)$ generates the two dimensional diffeomorphisms, but, since ξ^α depends on the worldsheet coordinates, this is a local invariance . Moreover, ϵ is a Majorana two dimensional spinor, like ψ^μ, but, dislike ψ^μ, it is a scalar under Lorentz spacetime transformations (it has no indices of spacetime).

Notice that the two dimensional supersymmetric transformations (133) map two dimensional scalars into two dimensional spinors and viceversa. The last equation of (133) tells that the gravitino χ_α is the gauge field of the two dimensional supersymmetry, just like $\xi^\alpha(\tau, \sigma)$ is the gauge field of the diffeomorphisms.

The action (125), though invariant under worldsheet diffeomorphisms, is not invariant under the supersymmetry (133). To have an action invariant also under the supersymmetry (133), we must add two more terms. They are

$$S_1 \, [X^\mu, \psi^\mu, h_{\alpha\beta}, \chi_\alpha] \; = \; -\frac{1}{\pi} \int d\tau d\sigma \, e \, \bar{\chi}_\alpha \, J^\alpha \; , \qquad J^\alpha \; = \; \frac{1}{2} \rho^\beta \rho^\alpha \, \psi^\mu \, \partial_\beta \, X_\mu \; , \tag{134}$$

$$S_2 \, [X^\mu, \psi^\mu, h_{\alpha\beta}, \chi_\alpha] \; = \; -\frac{1}{4\pi} \int d\tau d\sigma \, e \, \bar{\psi}^\mu \psi_\mu \, \bar{\chi}_\alpha \, \rho^\beta \rho^\alpha \chi_\beta \; . \tag{135}$$

Notice that

$$e \; = \; \sqrt{-h} \; , \tag{136}$$

while J^α is the supercurrent, i.e. just the Noether conserved current deriving from the invariance of the action under the two dimensional supersymmetry.

The term S_1 plays the same role of the gauge invariant term $A_\mu \, J^\mu$ in four dimensional QED.

Thus, we have that the complete action

$$S \, [X^\mu, \psi^\mu, h_{\alpha\beta}, \chi_\alpha] \; = \; S_0 + S_1 + S_2 \; , \tag{137}$$

is invariant under both the diffeomorphisms and the supersymmetric transformation (133). Such properties makes it the good starting point to study the supersymmetric string theory, i.e. the *superstring theory.*

The action (137) is invariant also under another set of fields transformations.

They are

$$\begin{cases} \delta \, X^\mu & = \; 0 \; , \\ \delta \, \psi^\mu & = \; -\dfrac{1}{2} \, \Lambda \, \psi^\mu \; , \\ \delta \, e^a_\alpha & = \; \Lambda \, e^a_\alpha \; , \\ \delta \, \chi_\alpha & = \; \dfrac{1}{2} \, \Lambda \, \chi_\alpha \; . \end{cases} \tag{138}$$

These are the *Weyl transformations* for our supersymmetric theory. We recall that the Weyl transformations do not change the worldsheet coordinates, but only the fields involved in the theory. The Weyl transformations just rescale the worldsheet metric without touching the other fields and they are not suspersymmetric transformation because, as one can immediately see from their infinitesimal expression (138), they map bosons in bosons and fermions in fermions.

From (138), we read that ψ^μ has Weyl weight $1/2$, e_α^a has Weyl weight -1 and χ_α has Weyl weight $-1/2$.

Finally, the action (137) has one last further invariance. The infinitesimal version of its transformations is

$$\begin{cases} \delta X^\mu &= 0 \,, \\ \delta \psi^\mu &= 0 \,, \\ \delta e_\alpha^a &= 0 \,, \\ \delta \chi_\alpha &= i\, \rho_\alpha \, \eta \,. \end{cases} \tag{139}$$

Here $\eta(\tau,\sigma)$ is another Majorana spinor, that we can use to eliminate ψ^+, for instance. These are the *infinitesimal superconformal transformations* and they are very special diffeomorphisms.

Now we are ready to study the classical theory defined by the supersymmetric action

$$\begin{aligned} S_0\,[\,X^\mu,\psi^\mu,h_{\alpha\beta}\,] \;=\; & -\frac{1}{2\pi}\int d\tau d\sigma\; e\,\left(h^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\eta_{\mu\nu} - i\,\bar{\psi}^\mu\rho^\alpha\nabla_\alpha\psi^\nu\eta_{\mu\nu}\right) \\ & -\frac{1}{\pi}\int d\tau d\sigma\; e\,\bar{\chi}_\alpha\,J^\alpha \\ & -\frac{1}{4\pi}\int d\tau d\sigma\; e\,\bar{\psi}^\mu\psi_\mu\,\bar{\chi}_\alpha\,\rho^\beta\rho^\alpha\chi_\beta \,. \end{aligned} \tag{140}$$

In the bosonic theory, we have chosen the light cone spacetime coordinates and in such coordinates system we have fixed the gauge in such a way that we have been left with only the physical degrees of freedom. The gauge fixing (55) for the bosonic theory in the [GSW] notation reads $X^+ = p^+\tau^{GSW}$ and the final action that one gets through this gauge fixing is (58), that in the [GSW] notation becomes

$$\begin{aligned} & S_P\,[\,X^\mu,\gamma_{ab}\,]\,|_{\,\text{gauge fixed}} \;= \\ & \quad -\frac{1}{2\pi}\int d\tau\int_0^\pi d\sigma\,\left(2\,p^+\,\partial_\tau X^- + \sum_{i=1}^{D-2}\left(-\,\partial_\tau X^i\,\partial_\tau X^i + \,\partial_\sigma X^i\,\partial_\sigma X^i\,\right)\right)\,. \end{aligned} \tag{141}$$