ROM2F-99/45

OPEN STRINGS

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U. Roma "Tor Vergata"

Andrejewski Lectures

(Berlin, Fall 1999)

1. Open Descendants, or Orientifolds
2. Surprising Features of Open-String Vacua
3. Lessons from and for Low-Energy Supergravity
4. Lessons from and for Boundary CFT

FOREWORD

These Lectures describe a research that started with my Caltech Ph.D. Thesis, carried out under the supervision of J.H. Schwarz.

After the early work done with Neil Marcus at Caltech and Berkeley, most of the work was carried out at the University of Rome "Tor Vergata", where I have had the opportunity to supervise the Ph.D research of M. Bianchi, G. Pradisi, C. Angelantonj and F. Riccioni, and to collaborate with Ya.S. Stanev, with us first as an INFN Fellow, then as guest of the Physics Department and, more recently, as a Senior Research Fellow. I owe much to them, as well as to the others with whom I have collaborated on these matters: D. Fioravanti, S. Ferrara, R. Minasian, I. Antoniadis, E. Dudas and G. D'Appollonio. Without their efforts and their enthusiasm, this work could not have been done.

It is a pleasure to stress, in particular, how M. Bianchi and G. Pradisi have played a crucial role in the early developments described in the second lecture, that lay the foundations for all the rest and how, somewhat later, Ya.S. Stanev has played a similar role in the formal developments described in the last lecture.

Finally, I would like to thank the Institutions where I have have presented these Lectures so far: the University of Torino (Fall 1998), where I was kindly invited by P. Fré and S. Sciuto, the LPT-Orsay (Spring 1999), where I was kindly invited by P. Binétruy and E. Dudas, and lastly the Humboldt University of Berlin (Fall 1999), where I was kindly invited by D. Lüst and received the kind and generous support of the Andrejewski Foundation.

Tor Vergata, November 1999 Augusto Sagnotti

C O N T E N T S

1. Open Descendants, or Orientifolds

a. Gravity from strings

b. Boundaries and crosscaps in the Polyakov series

c. Chan-Paton groups and “quarks” at the ends

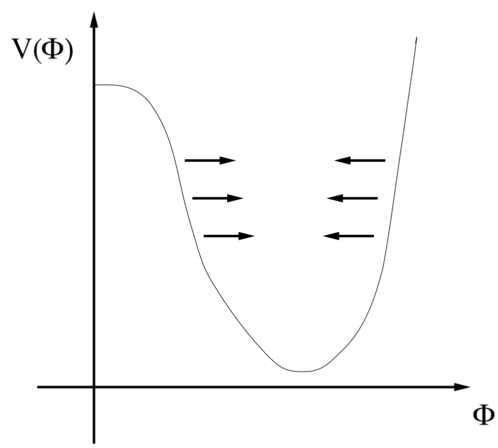
d. The closed bosonic string

e. The SO(8192) open bosonic string

f. Tadpoles

Consider an n-component real vector of scalars Φ, with

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \Phi^T \cdot \partial^\mu \Phi \; - \; \lambda \big(\Phi^T \cdot \Phi - \rho^2 \big)^2 \right)$$



Narrowing the mexican hat

The **formal** limit $\lambda \to \infty$ gives

$$S = \int d^4x \, \frac{1}{2} \partial_\mu \Phi^T \cdot \partial^\mu \Phi$$

but now Φ parameterizes the sphere

$$\Phi^T \cdot \Phi = \rho^2$$

The resulting σ-model is geometrical but non renormalizable: the radial oscillation is **"frozen"**.

Gravity from strings

Einstein's theory is a non-renormalizable and geometrical theory of space-time dynamics:

What are its frozen modes?

STRINGS, PARTICLES and P-BRANES

What is left from Riemannian geometry?

Harder question: WE DO NOT KNOW

Gravity from strings

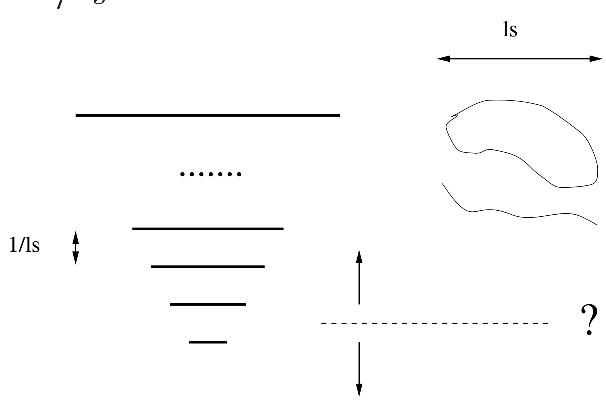
Gravitational interactions are **"hard"** at high energies. The effective fine structure function

$$\alpha_G = \frac{G_N E^2}{\hbar c^5}$$

becomes of order unity at the Planck scale:

$$E_{Pl} = \sqrt{\frac{\hbar c^5}{G_N}} \sim 10^{19}\ GeV$$

STRINGS: one-dimensional objects with a characteristic size l_s. They appear **point-like** at distance scales $r \gg l_s$, but have an infinity of (particle-like) modes of increasing masses, $m \sim 1/l_s$:



The string scale determines the string spectrum. Quantum effects fix the ground state energy.

String interactions become **"soft"** at the string scale.

There are **different types** of string theories: all their spectra include a massless spin-2 mode.

FROZEN MODES of EINSTEIN GRAVITY ?

Space-time Supersymmetry (Bose $\leftrightarrow$ Fermi) implies a stable vacuum ($\Delta E_{Bose} + \Delta E_{Fermi} = 0$).

(Supersymmetry breaking (typically) produces corrections to the vacuum energy.)

SUPERSTRING THEORIES: live in **10D space time**, and can be connected to our **4D world** by the Kaluza-Klein mechanism.

Two settings for the Kaluza-Klein mechanism:

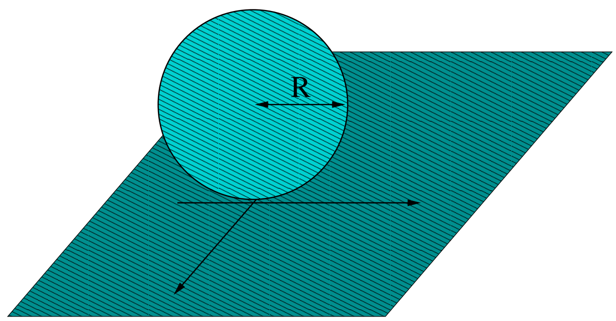
1) Both matter and gravity live in $D = 10$;

2) Gravity lives in $D = 10$ but matter may be confined to **lower-dimensional "islands"** (branes).

Gravity from strings

KALUZA-KLEIN: Space-time symmetry breaking. Vacuum state has a reduced (space-time) symmetry.

Reduction from $D+1$ to D dimensions on a circle of radius R:



The Kaluza-Klein mechanism

Ordinary Kaluza-Klein reduction $(D+1) \to D$, with co-ordinates (x^μ, y):

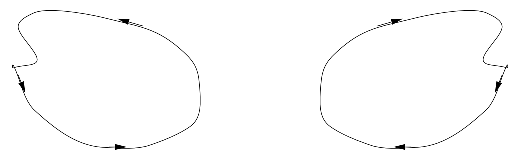
$$\nabla^2_{(D+1)} = \nabla^2_D - (\partial_y)^2$$

Harmonic analysis yields **mass eigenstates** $(m \sim 1/R)$.

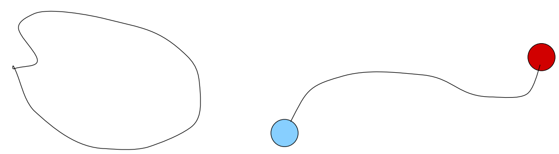
Low-energy modes : **zero modes** of internal Laplacian.

Ten-dimensional superstrings

The figure displays the different types of 10D strings:



Oriented closed strings: Type IIA, Type IIB, Heterotic



Unriented closed and open strings: Type I

Types of 10D superstrings

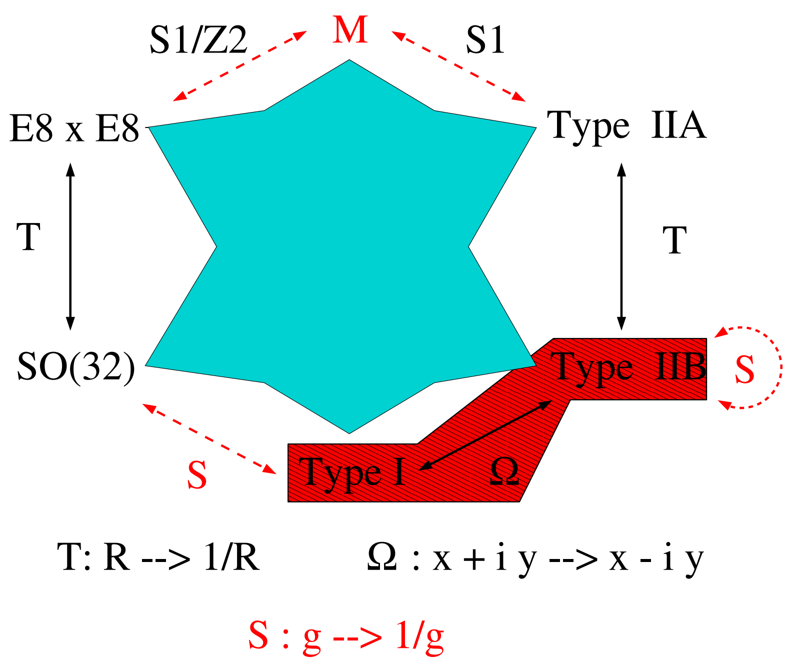
They differ both in the nature of the strings and in the nature of their long-range space-time fields (and even in the number of supersymmetries)

Different ways to relax the "frozen" modes of gravity?

No!

All 5 models (and more) are related by String Dualities.

Duality relations



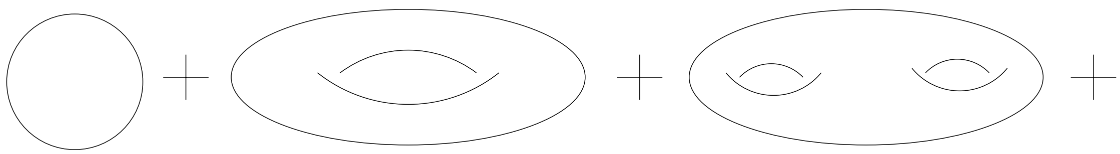
- Link the 5 10D strings to the 11D "M theory", while establishing **relations** between corresponding **moduli** (undetermined vacuum values).

- String descriptions of low-energy supergravity differ by **field redefinitions**.

- Here we shall focus on the relation between IIB and I and on its generalizations.

The Polyakov series

- Two types of strings: **closed** and **open**.
- Two types of each: **oriented** and **unoriented**.
- Open strings: **Chan-Paton charges** at ends.



- Perturbative series (**oriented closed** strings): Riemann surfaces, **closed** and **orientable**, ordered by

$$g_s^{-\chi} = g_s^{(2h-2)} = \left(e^{<\phi>}\right)^{2h-2}$$

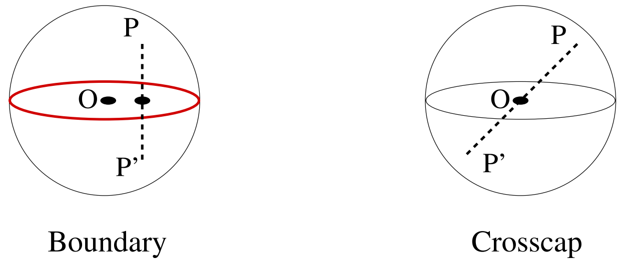
where χ is the Euler character, h is the number of **handles** and ϕ is a ubiquitous scalar, the **dilaton**.

- **Open descendants:** surfaces with also (b) boundaries and (c) crosscaps ($g_s^{(2h+b+c-2)}$), but:

$$3c \equiv 1h + 1c$$

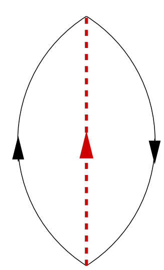
Boundaries and crosscaps

- All surfaces with boundaries and crosscaps are **doubly covered** by closed and orientable ones.



- We can use this trick to visualize a **boundary** via the double covering of a disk, a sphere: the boundary is the **equator**.

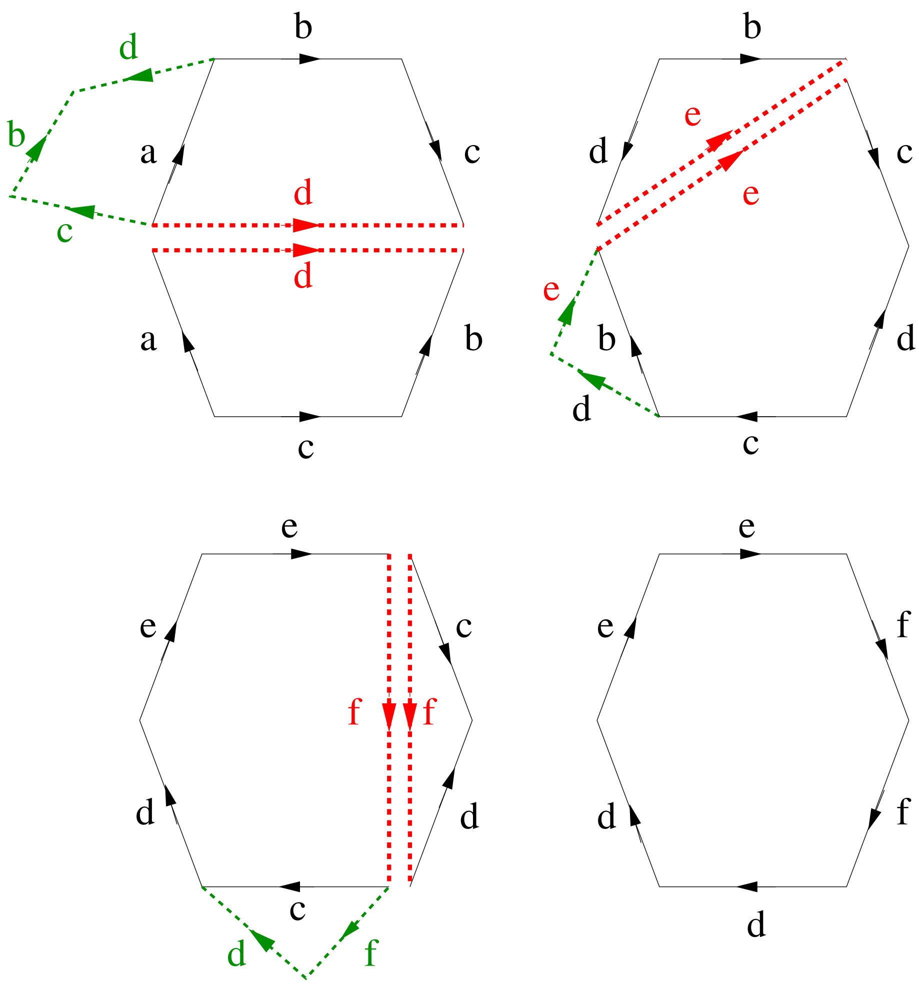
- We can also **define** the **crosscap**, identifying antipodal points. Here we find the **real projective plane**, a **closed non-orientable** surface: a disk with pair-wise identifications of opposite points on the boundary.



- **Twice** the vertical path is **contractible**.

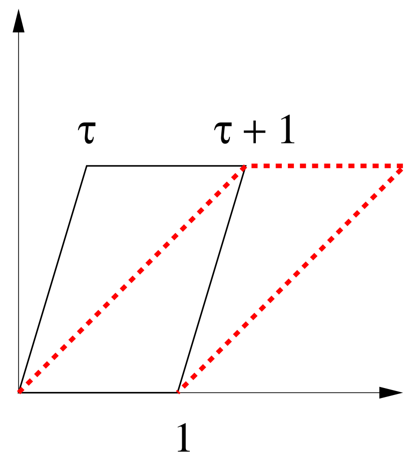
$3c \equiv 1h + 1c$

- One can dissect surfaces with suitable number of cuts.



- **1 handle** and **1 crosscap** can be turned into **3 crosscaps** with the 3 moves in the figure.

Surfaces with vanishing Euler character



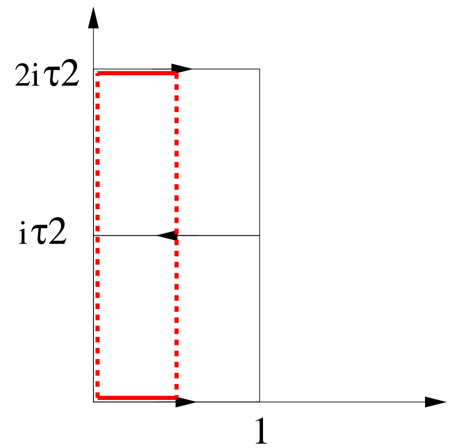
1. The **torus** can be dissected into the $(1, \tau)$ parallelogram with opposite sides identified. The basic datum is actually a **2D lattice**: infinitely many equivalent choices of parallelograms of **minimal area** related by:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad (a, b, c, d) \in Z \ , \quad ad - bc = 1$$

- This is sufficient to describe the perturbative spectra of **oriented closed** strings.

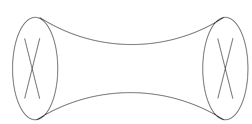
- For open descendants one must also consider the **Klein bottle**, the **annulus** and the **Möbius strip**.

The Klein bottle



2. The **Klein bottle** can be dissected into the $(1, i\tau_2)$ rectangle. Now the horizontal sides are identified only after a relative **reversal**.

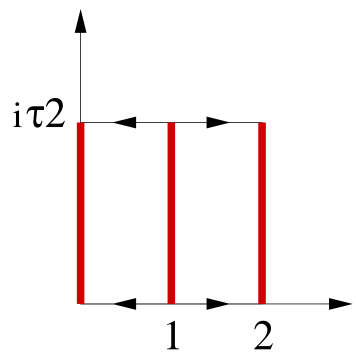
- The doubly covering torus is the $(1, 2i\tau_2)$ rectangle, **of modulus** $2i\tau_2$.



- An alternative choice of fundamental region (red) exhibits it as a tube terminating at two **crosscaps** (dashed vertical lines).

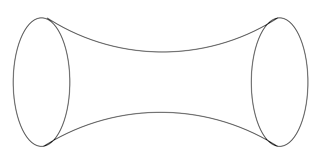
- **Notice:** vertical UV $\leftrightarrow$ horizontal IR.

The annulus

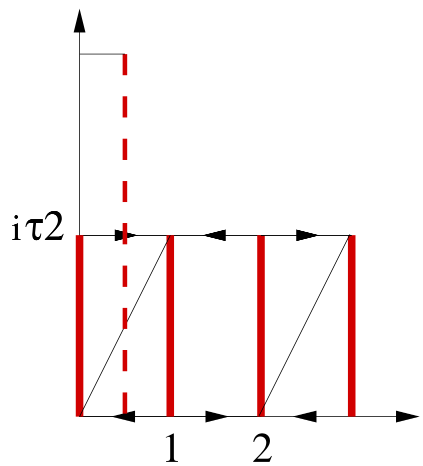


3. The **annulus** can be dissected into the $(1, i\tau_2)$ rectangle, where the vertical sides are the **boundaries**.

- The doubly covering torus is the $(2, i\tau_2)$ rectangle, **of modulus** $i\tau_2/2$.

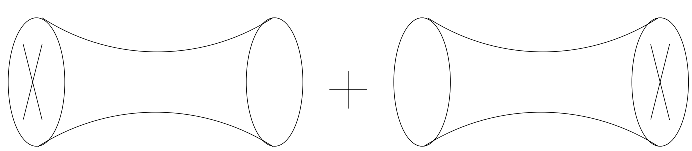


- **Notice:** vertical UV $\leftrightarrow$ horizontal IR.



4. The **Möbius strip** can be dissected into the $(1, i\tau_2)$ rectangle, where the vertical sides are (two halves of) the **boundary** and the horizontal sides have **opposite** orientations.

- The doubly covering torus is **skew**, **of modulus** $i\tau_2/2 + 1/2$.
- An alternative choice of fundamental region (red) exhibits it as a tube terminating at one boundary and one **crosscap** (dashed vertical line).



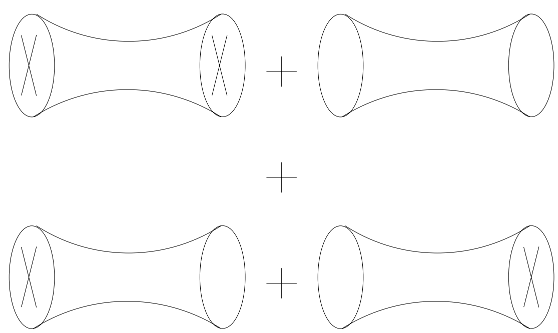
The tadpole condition

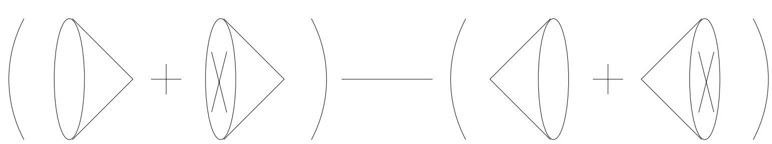
- In String Theory, the modular integration for the torus is restricted to a fundamental region, away from the UV limit.

- This is not true for the other surfaces, where the integration line touches the real (τ_1) axis.

- **However:** with a suitable (horizontal) choice of time, all other three surfaces describe the propagation of **closed** strings.

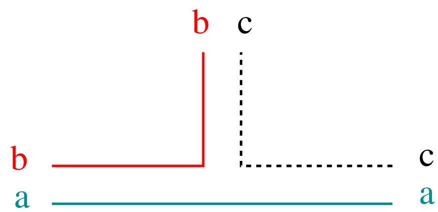
- The resulting IR singularities can be eliminated by combining the three additional contributions:





Chan-Paton charges

- One can associate pairs of **charges** to the **ends** of open strings. This leads to an effective "coloring" of boundaries.



- One can thus associate **matrices** to the **amplitudes** of the now group-valued open strings.

(Chan and Paton, 1969)

- This symmetry becomes a **gauge symmetry** in String Theory. One can build in this way **all** the **classical** groups $O(n)$, $U(n)$ and $USp(2n)$, but **not** the **exceptional** groups.

(Schwarz, 1982)
(Marcus and A.S., 1982)

- One can also recover this setting from **1D fermions** living at the ends of strings, also useful to compute amplitudes in external backgrounds.

(Marcus and A.S., 1986)
(Dorn and Otto, 1986)

The bosonic string: light-cone quantization

- The action for a bosonic string (in flat space time) describes 2D scalars coupled to 2D gravity:

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \, \partial_\beta X^\nu \eta_{\mu\nu} + \frac{\langle \phi \rangle}{2\pi} \int d^2\xi \sqrt{-g} R$$

- The conformal gauge $g_{\alpha\beta} = \eta_{\alpha\beta}\Lambda$ leads to a free wave equation, and (at least in a **critical dimension**) Λ disappears from the functional measure.

- For the **closed** string:

$$X^\mu = x^\mu + (2\alpha')p^\mu \, \tau + \frac{i\sqrt{2\alpha'}}{2} \sum_{n\neq 0} \left(\frac{\alpha_n^\mu}{n} \, e^{-2in(\tau-\sigma)} + \text{“tilde”} \right)$$

- For the **open** string:

$$X^\mu = x^\mu + (2\alpha')p^\mu \, \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} \, e^{-in\tau} \cos(n\sigma)$$

- An infinite (**conformal**) symmetry is left, and allows one to choose the **light-cone gauge**:

$$X^+ = x^+ + (2\alpha')p^+ \, \tau$$

The bosonic string: light-cone quantization

- The field equations of $g_{\alpha\beta}$ relate X^- to the transverse modes:

$$(2\alpha')p^+ \, \partial_\pm X^- \; - \; (\partial_\pm X^i)^2 = 0$$

- These constraints define the (transverse) **Virasoro operators** (and, for the closed string, also corresponding "tilde" operators):

$$L_m = \frac{1}{2} : \sum_n \, \alpha^i_{m-n} \alpha^i_n :$$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{(D-2)}{12} \, m(m^2-1) \delta_{m+n,0}$$

- The **closed** mass spectrum is defined by:

$$M^2 = \frac{2}{\alpha'} \left(L_0 + \bar{L}_0 - \frac{D-2}{12} \right) \qquad L_0 = \bar{L}_0$$

- The **open** mass spectrum is defined by:

$$M^2 = \frac{1}{\alpha'} \left(L_0 - \frac{D-2}{24} \right)$$

- **Lorentz invariance** requires $D = 26$.

Vacuum energy in Field Theory

- In **String Theory** the **vacuum energy** places important restrictions on the (perturbative) spectrum.

- Start from **Field Theory**, considering a (Euclidean) **scalar** of mass M:

$$e^{-\Gamma} = \int [D\phi] e^{-S_E} \sim \det{}^{-\frac{1}{2}} \left(-\Delta + M^2\right)$$

- The M dependence of Γ may be extracted from:

$$\log\left(\det(A)\right) = - \int_{\epsilon}^{\infty} \frac{dt}{t} \, \mathrm{tr}\left(e^{-tA}\right)$$

- The result is:

$$\Gamma = \frac{V}{2(4\pi)^{D/2}} \int_{\epsilon}^{\infty} \frac{dt}{t^{D/2+1}} \, e^{-tM^2}$$

- For a general model involving **bosons** and **fermions**:

$$\Gamma_{tot} = \frac{V}{2(4\pi)^{D/2}} \int_{\epsilon}^{\infty} \frac{dt}{t^{D/2+1}} \, \mathrm{Str}\left(e^{-tM^2}\right)$$

- Str : bosons and fermions with **opposite signs**.

Vacuum energy for oriented closed strings

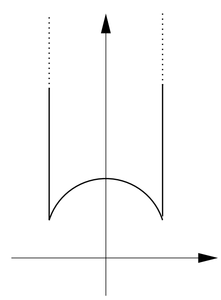
- For oriented closed strings (**naively**):

$$\Gamma = \frac{V}{2(4\pi)^{13}} \int_{-\infty}^{\infty} ds \int_{\epsilon}^{\infty} \frac{dt}{t^{14}} \text{ tr } \left(e^{-\frac{2}{\alpha'}(L_0+\bar{L}_0-2)t} e^{2\pi i (L_0-\bar{L}_0)s} \right)$$

- Defining

$$\tau = \tau_1 + i\tau_2 = s + i\frac{t}{\alpha'\pi} \qquad q = e^{2\pi i \tau} \qquad \bar{q} = e^{-2\pi i \bar{\tau}}$$

$$\Gamma = \frac{V}{2(4\pi^2\alpha')^{13}} \int_{-\infty}^{\infty} d\tau_1 \int_{\epsilon}^{\infty} \frac{d\tau_2}{\tau_2^{14}} \text{ tr } \left(q^{L_0-1} \bar{q}^{\bar{L}_0-1} \right)$$



- Actually, the **inequivalent** tori correspond to a UV free **fundamental region** $\mathcal{F}$ of $SL(2,Z)$, and (rescaling):

$$\mathcal{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{\tau_2^{12}} \text{ tr } \left(q^{L_0-1} \bar{q}^{\bar{L}_0-1} \right)$$

Vacuum energy for the bosonic string

- One can compute the traces using standard Bose gas tricks:

$$L_0 = \sum_k \; k \; a_k^\dagger \; a_k$$

$$\text{tr} \; q^{k \, a_k^\dagger a_k} = 1 + q^k + q^{2k} + \ldots = \frac{1}{1 - q^k}$$

$$\mathcal{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \, \frac{1}{\tau_2^{12}} \, \frac{1}{|\eta(\tau)|^{48}}$$

- The Dedekind η function

$$\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

transforms under the two generators of $SL(2, Z)$ as:

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \, \eta(\tau) \qquad \eta(-1/\tau) = \sqrt{-i\tau_2} \; \eta(\tau)$$

- The integration measure is **modular invariant**.

- The contribution $\sqrt{\tau_2}|\eta(\tau)|^2$ of **each** string coordinate is **modular invariant**.

Open descendants: $\mathcal{K}$ projection

- We want to **project** the closed spectrum, keeping only states **symmetric** under $\Omega : \alpha \leftrightarrow \tilde{\alpha}$.

- At the two **lowest levels** :

  tachyon : $|0\tilde{0}>$ graviton and dilaton : $\alpha^{(i}_{-1}\,\tilde{\alpha}^{j)}_{-1}|0\tilde{0}>$

$$1 \to 1 \qquad (24)^2 \to \frac{24(24+1)}{2} \quad \text{states}$$

- The torus partition function (actually, its terms with equal powers of q and $\bar{q}$) **counts** the original closed spectrum. To project:

$$\mathcal{T} \to \frac{1}{2}(\mathcal{T} + \mathcal{K})$$

- K **counts** all states **fixed** under Ω:

$$\sum_{\mathrm{L,R}} \langle \mathrm{L,R}|\ q^{L_0-1}\ \bar{q}^{\bar{L}_0-1}\ |\mathrm{R,L}\rangle = \sum_{\mathrm{L}} \langle \mathrm{L,L}|\ (q\bar{q})^{L_0-1}\ |\mathrm{L,L}\rangle$$

- For the bosonic string:

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{14}}\,\frac{1}{\eta^{24}(2i\tau_2)}$$

Open descendants: $\tilde{\mathcal{K}}$, $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{M}}$

- By a **redefinition** $\tau_2 \to t/2$ and an S **transformation** $t \to 1/\ell$ we can turn $\mathcal{K}$ to $\tilde{\mathcal{K}}$, the "crosscap-to-crosscap" amplitude:

$$\tilde{\mathcal{K}} = \frac{2^{13}}{2} \int_0^\infty d\ell \, \frac{1}{\eta^{24}(i\ell)}$$

- Now we can get the "boundary-to-boundary" amplitude $\tilde{\mathcal{A}}$ letting the **closed** spectrum flow in the tube terminating at two boundaries (what boundaries, in general CFT?).

$$\tilde{\mathcal{A}} = \frac{N^2 \, 2^{-13}}{2} \int_0^\infty d\ell \, \frac{1}{\eta^{24}(i\ell)}$$

- N is a **reflection coefficient**.

- $\tilde{\mathcal{M}}$, the "boundary-to-crosscap" amplitude, is determined by the **"geometric mean"** of $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{A}}$, and **depends on** $i\ell + 1/2$:

$$\tilde{\mathcal{M}} = 2 \, \frac{N}{2} \int_0^\infty d\ell \, \frac{1}{\hat{\eta}^{24}(i\ell + 1/2)}$$

$$\left[\; \hat{\eta}(i\ell + 1/2) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - (-1)^n q^n) \; \right]$$

The open spectrum

- We can recover $\mathcal{A}$ by an S^{-1} **transformation** $\ell \rightarrow 1/t$ and a **redefinition** $t \rightarrow t/2$:

$$\mathcal{A} = \frac{N^2}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{14}} \frac{1}{\eta^{24}\left(\frac{i\tau_2}{2}\right)}$$

- We can also recover $\mathcal{M}$ by a **redefinition** $\ell = t/2$ and a P^{-1} **transformation**:

$$P^{-1} \;:\; \frac{it}{2} + \frac{1}{2} \rightarrow \frac{i}{2\tau_2} + \frac{1}{2}$$

- $\mathcal{A}$ and $\mathcal{M}$ **count** the **open** spectrum.

- At the two **lowest levels** :

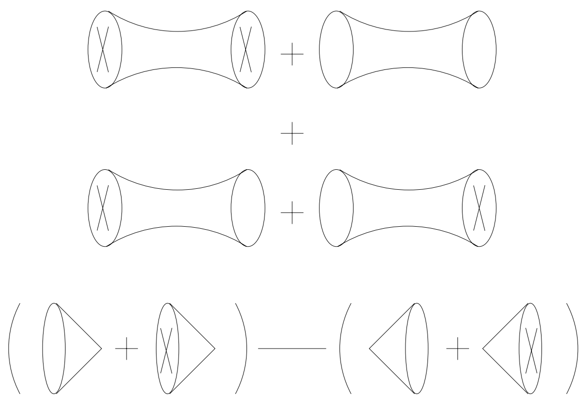
$$\frac{N(N+1)}{2}\ \text{tachyons} \qquad \frac{N(N-1)}{2}\text{vectors}$$

- Can we fix the **(signed) integer** N?

Tadpoles

Dilaton tadpole and $SO(8192)$

(Douglas and Grinstein, 1986)
(Marcus and A.S., 1986)
(Weinberg, 1986)



- The vacuum-channel contains integrals like:

$$\int_0^\infty d\ell e^{-M^2\ell} = \frac{1}{M^2} = \left(\frac{1}{p^2+M^2}\right)_{p=0}$$

  singular for massless modes. One can arrange for cancellations. Here the singularity is proportional to

$$2^{13} + \frac{N^2}{2^{13}} - 2N = 2^{-13}\left(N - 2^{13}\right)^2$$

- $N = 2^{13} = 8192$ **eliminates** a dilaton potential.

- **More importantly**: in theories with chiral fermions some **tadpoles** are related to **anomalies**.

(Polchinski and Cai, 1988)

Summary:Open descendants, or orientifolds

1. Start from an **oriented** closed string with a Z_2 symmetry Ω interchanging left and right modes.

2. **Project** the closed spectrum using Ω :

$$\mathcal{T} \to \frac{1}{2}(\mathcal{T} + \mathcal{K})$$

3. **Derive** $\tilde{\mathcal{K}}$ from $\mathcal{K}$ via an S transformation.

4. Obtain $\tilde{\mathcal{A}}$ from the transverse-channel propagation of all **allowed** sectors of the **closed** spectrum.

5. **Derive** $\mathcal{A}$ from $\tilde{\mathcal{A}}$ via an S^{-1} transformation.

6. **Determine** $\tilde{\mathcal{M}}$ from $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{A}}$.

7. **Derive** $\mathcal{M}$ from $\tilde{\mathcal{M}}$ via a P^{-1} transformation.

$$P = T^{1/2} S T^2 S T^{1/2}$$

2. Surprising properties of open-string vacua

a. Ten-dimensional models and symmetry breaking

b. Exotic Klein-bottle projections

c. Toroidal compactification and quantized B_{ab}

d. Orbifold models

e. Brane supersymmetry (breaking)

The NSR superstring

- Now move to the NSR **superstring**

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-g}\Big(g^{\alpha\beta}\partial_\alpha X^\mu\ \partial_\beta X^\nu\ \eta_{\mu\nu}$$
$$+\ i\bar{\lambda}^\mu\gamma^\alpha\nabla_\alpha\lambda^\nu\eta_{\mu\nu} + i\bar{\psi}_\alpha\gamma^\beta\gamma^\alpha\lambda^\mu(\partial_\beta X^\nu - \frac{i}{4}\bar{\psi}_\beta\lambda^\nu)\ \eta_{\mu\nu}\Big)$$

- As for the bosonic string, in the **critical dimension** ($D = 10$) this reduces to a free theory, with:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\Big(\partial^\alpha X^\mu\ \partial_\alpha X^\nu\ \eta_{\mu\nu} + i\bar{\lambda}^\mu\gamma^\alpha\partial_\alpha\lambda^\nu\ \eta_{\mu\nu}\Big)$$

- Again, constraints set to zero the **e.m. tensor** and now also the **supercurrent**. Using the remaining symmetry, let:

$$X^+ = x^+ + (2\alpha')p^+\tau \qquad , \qquad \lambda^+ = 0$$

- Again, only **transverse** excitations are left.

- The Noether currents of the Poincaré group are even in the fermions, that can be (anti)periodic along a closed string, giving the (NS)R sectors.

The GSO projection

- This gives **four** sectors ($NS-NS$,$R-R$,$NS-R$, $R-NS$) for **closed** strings and **two** sectors (NS,R) for **open** strings, where $\dot{X} \pm X'$ depend on $(\tau \pm \sigma)$.

- The spectrum is built by **modes** of X and λ, but the latter are **anti-commuting**.

- **Consistent statistics** for string excitations: even and odd powers of λ should be treated differently.

  (Gliozzi, Scherk and Olive, 1977)

- The NS vacuum is a **singlet**, while the R vacuum is a **Majorana spinor**, that can be split into a **pair** of **Majorana-Weyl** spinors: $|S> \rightarrow |L> + |R>$.

- We can build the corresponding **partition functions** using elementary properties of the **Fermi gas**.

Partition functions

(Brink and Nielsen, 1973)

- The contributions to the **vacuum-energy shift** are:

  -1/24 for **periodic bosons**
  +1/24 for **periodic fermions**
  -1/48 for **anti-periodic fermions**

- Aside from the vacuum shift, the fermion contribution is:

$$tr\left(q^{\sum_r r\lambda_r^\dagger \lambda_r}\right) = \prod_r tr\left(q^{r\lambda_r^\dagger \lambda_r}\right) = \prod_r (1+q^r)^8$$

- r is 1/2-**odd integer** for the $\mathbf{NS}$ sector and **integer** for the $\mathbf{R}$ sector.

- The full **NS sector** contribution from **odd** numbers of fermionic oscillators (8 **periodic bosons** + 8 **antiperiodic fermions**) is:

$$\frac{\prod_{m=1}^{\infty}(1+q^{m-1/2})^8 - \prod_{m=1}^{\infty}(1-q^{m-1/2})^8}{2\ q^{1/2}\prod_{m=1}^{\infty}(1-q^m)^8}$$

- **No tachyon**, the spectrum starts with a massless vector.

Partition functions and characters

- A **better notation** uses level-one $so(8)$ **characters**:

$$\frac{O_8}{\eta^8} = \frac{\prod_{m=1}^{\infty}(1+q^{m-1/2})^8 + \prod_{m=1}^{\infty}(1-q^{m-1/2})^8}{2\ q^{1/2}\prod_{m=1}^{\infty}(1-q^m)^8}$$

$$\frac{V_8}{\eta^8} = \frac{\prod_{m=1}^{\infty}(1+q^{m-1/2})^8 - \prod_{m=1}^{\infty}(1-q^{m-1/2})^8}{2\ q^{1/2}\prod_{m=1}^{\infty}(1-q^m)^8}$$

- In terms of **Jacobi** ϑ **functions**:

$$O_8 = \frac{\vartheta_3^4 + \vartheta_4^4}{2\eta^4}$$

$$V_8 = \frac{\vartheta_3^4 - \vartheta_4^4}{2\eta^4}$$

- There are two more characters for the spinor classes:

$$S_8 = \frac{\vartheta_2^4 + \vartheta_1^4}{2\eta^4}$$

$$C_8 = \frac{\vartheta_2^4 - \vartheta_1^4}{2\eta^4}$$

- There is an **ambiguity**, since $\vartheta_1 \equiv 0$, while :

$$\frac{\vartheta_2^4}{\eta^{12}} = 16\ \frac{\prod_{m=1}^{\infty}(1+q^m)^8}{\prod_{m=1}^{\infty}(1-q^m)^8}$$

Ten-dimensional NSR models: S, T and P

- For $\mathcal{K}$ and $\mathcal{A}$, the **direct** and **transverse** channels are related by:

$$S : \tau \to -\frac{1}{\tau}$$

- For $\mathcal{M}$ one needs an **additional transformation**:

$$P : i\frac{\tau_2}{2} + \frac{1}{2} \to i\frac{1}{2\tau_2} + \frac{1}{2}$$

- One can show that $P = T^{1/2} S T^2 S T^{1/2}$ for the **real basis** of characters $\hat{\chi}$ ($q = e^{-2\pi t}$):

$$\chi = q^{h-c/24} \sum_n d_n q^n \to \hat{\chi} = q^{h-c/24} \sum_n (-1)^n d_n q^n$$

- In our case one can **resolve** the ambiguity, defining **matrices** S, T and P that implement on $(O, V, S, C)/\eta^8$ the corresponding modular transformations:

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Ten-dimensional NSR models

- Several NSR models of **oriented closed** strings, determined by **modular invariance**, a **completeness condition** for the set of closed string (bulk) operators:

1. **Type-IIA**, with the **massless** modes (e^a_μ, ϕ, A_μ, $B_{\mu\nu}$, $C_{\mu\nu\rho}$; ψ_μ, λ):

$$\mathcal{T}_{IIA} = (\bar{V}_8 - \bar{S}_8)(V_8 - C_8)$$

2. **Type-IIB**, with the **massless** modes (e^a_μ, ϕ, ϕ', $B_{\mu\nu}$, $B'_{\mu\nu}$, $D^{(+)}_{\mu\nu\rho\sigma}$; $\psi_{\mu,R}$, λ_L, $\psi'_{\mu,R}$, λ'_L)

$$\mathcal{T}_{IIB} = |V_8 - S_8|^2$$

3. **Type-0A**, with the **massless** modes (e^a_μ, ϕ, $B_{\mu\nu}$, A_μ, A'_μ, $C_{\mu\nu\rho}$, $C'_{\mu\nu\rho}$) and a **tachyon** T:

$$\mathcal{T}_{0A} = |O_8|^2 + |V_8|^2 + \bar{S}_8 C_8 + \bar{C}_8 S_8$$

4. **Type-0B**, with the **massless** modes (e^a_μ, ϕ, ϕ', ϕ'', $B_{\mu\nu}$, $B'_{\mu\nu}$, $B''_{\mu\nu}$, $D^{(+)}_{\mu\nu\rho\sigma}$, $D^{(-)}_{\mu\nu\rho\sigma}$) and a **tachyon** T:

$$\mathcal{T}_{0B} = |O_8|^2 + |V_8|^2 + |S_8|^2 + |C_8|^2$$

[3 and 4] : (Dixon and Harvey; Seiberg and Witten, 1985)

Ten-dimensional NSR models

- According to **Jacobi's** *aequatio*

$$V_8 = S_8 = C_8$$

  the V, S and C classes contain equal number of states at all levels, a reflection of the (two) **space-time supersymmetries** of type-IIA and type-IIB.

- Like the **bosonic** string, IIB, OA and OB are **symmetric** under $L \leftrightarrow R$ interchange.

- The (supersymmetric) **open descendant** of IIB is the $SO(32)$ **type-I superstring** of Green and Schwarz.

  (Green and Schwarz, 1984)

- It is very interesting, but its construction adds **no new ingredients** to what we have seen for the bosonic $SO(8192)$ string. Therefore, in the following page we can just **list** the corresponding amplitudes $\mathcal{K}$, $\mathcal{A}$, $\mathcal{M}$, $\tilde{\mathcal{K}}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{M}}$:

Type-IIB $\to$ Type-I

- **Direct-channel** amplitudes:

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\,\frac{(V_8 - S_8)(2i\tau_2)}{\eta^8(2i\tau_2)}$$

$$\mathcal{A} = \frac{N^2}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\,\frac{(V_8 - S_8)(i\tau_2/2)}{\eta^8(i\tau_2/2)}$$

$$\mathcal{M} = -\frac{N}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\,\frac{(\hat{V}_8 - \hat{S}_8)(i\tau_2/2 + 1/2)}{\hat{\eta}^8(i\tau_2/2 + 1/2)}$$

- **Transverse-channel** amplitudes:

$$\tilde{\mathcal{K}} = \frac{2^5}{2}\int_0^\infty d\ell\,\frac{(V_8 - S_8)(i\ell)}{\eta^8(i\ell)}$$

$$\tilde{\mathcal{A}} = \frac{2^{-5}\ N^2}{2}\int_0^\infty d\ell\,\frac{(V_8 - S_8)(i\ell)}{\eta^8(i\ell)}$$

$$\tilde{\mathcal{M}} = -2\,\frac{N}{2}\int_0^\infty d\ell\,\frac{(\hat{V}_8 - \hat{S}_8)(i\ell + 1/2)}{\hat{\eta}^8(i\ell + 1/2)}$$

- **Tadpole condition**:

$$\frac{2^5}{2} + \frac{2^{-5}\ N^2}{2} - 2\,\frac{N}{2} = \frac{2^{-5}}{2}\,(N - 32)^2 = 0$$

Open descendants of $0A$

(Bianchi and A.S., 1990)

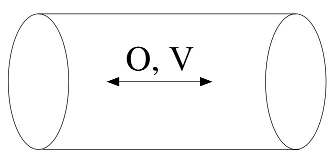
- Here the situation is more interesting:

$$\mathcal{K} = \frac{1}{2}(O_8 + V_8)$$
$$\tilde{\mathcal{K}} = \frac{2^5}{2}(O_8 + V_8)$$

- **Projected** closed spectrum: T, e_μ^a, ϕ, A_μ, $C_{\mu\nu\rho}$.

- $\tilde{\mathcal{K}}$ is obtained by the rescaling $2\tau_2 \to t$ and the S transformation $t \to 1/\ell$.

- $\tilde{\mathcal{A}}$ **can** contain the **two** sectors, O_8 and V_8, joined to their conjugates by the closed-string GSO, since $D = 10$ boundaries **must** respect the bulk symmetry.



- With a nice parameterization of the two **reflection** coefficients:

$$\tilde{\mathcal{A}} = \frac{2^{-5}}{2}\left((n_b + n_f)^2 V_8 + (n_b - n_f)^2 O_8\right)$$

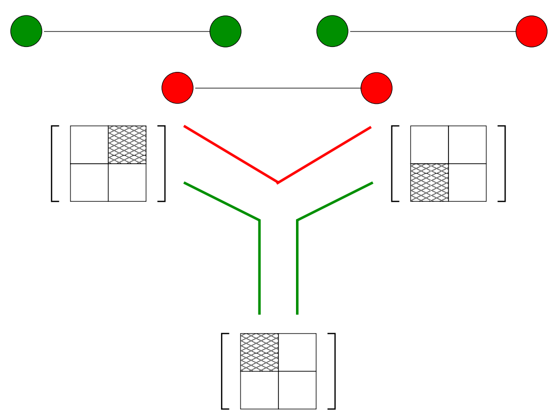
Open descendants of $0A$

- An S transformation $\ell \to 1/t$ and a rescaling $t \to \tau_2/2$ give $\mathcal{A}$:

$$\mathcal{A} = \frac{n_b^2 + n_f^2}{2}(O_8 + V_8) - n_b n_f (S_8 + C_8)$$

- What is the meaning of $\mathcal{A}$?

- **Three** types of open strings, $n_b - n_b$, $n_b - n_f$, $n_f - n_f$, corresponding to **two** types of Chan-Paton labels: **Chan-Paton symmetry breaking**.



- Some Chan-Paton matrices are **depleted** and their charges are **moved** to other sectors of the open spectrum, **consistently** with **interactions** [fusion].

Open descendants of $0A$

- How about the **Möbius** projection? We can **derive** $\tilde{\mathcal{M}}$ as a "geometric mean" of $\tilde{\mathcal{K}}\ \tilde{\mathcal{A}}$:

$$\begin{aligned}\tilde{\mathcal{K}} &= \frac{2^5}{2}(O_8+V_8)\\ \tilde{\mathcal{A}} &= \frac{2^{-5}}{2}\left((n_b+n_f)^2V_8+(n_b-n_f)^2O_8\right)\\ \tilde{\mathcal{M}} &= -\frac{2}{2}\left[(n_b+n_f)\hat{V}_8+(n_b-n_f)\hat{O}_8\right]\end{aligned}$$

- Now we can obtain $\mathcal{M}$ by a P transformation. The **open** spectrum is then:

$$\begin{aligned}\mathcal{A} &= \frac{n_b^2+n_f^2}{2}(O_8+V_8)-n_bn_f(S_8+C_8)\\ \mathcal{M} &= -\frac{1}{2}\left[(n_b+n_f)\hat{V}_8-(n_b-n_f)\hat{O}_8\right]\end{aligned}$$

- The gauge group is $SO(n_b)\times SO(n_f)$, and the spectrum is **not chiral**.
  There is a **dilaton tadpole** unless:

$$n_b+n_f=32$$

Open descendants of $0B$

(Bianchi and A.S., 1990)

- These are apparently harder to build, but actually have a **simpler structure**.

- Now **all** sectors flow in $\mathcal{K}$, and let us begin with the "natural" choice for the projection:

$$\mathcal{K}_1 = \frac{1}{2}(O_8 + V_8 - S_8 - C_8)$$

$$\tilde{\mathcal{K}}_1 = \frac{2^6}{2} V_8$$

- **Projected** closed spectrum: T, e^a_μ, ϕ, $B'_{\mu\nu}$, $B''_{\mu\nu}$.

- **All** sectors flow also in $\tilde{\mathcal{A}}$:

$$\tilde{\mathcal{A}}_1 = \frac{2^{-6}}{2}\Big[(n_o+n_v+n_s+n_c)^2 V_8 + (n_o+n_v-n_s-n_c)^2 O_8 - (-n_o+n_v+n_s-n_c)^2 S_8 - (-n_o+n_v-n_s+n_c)^2 C_8\Big]$$

$$\mathcal{A}_1 = \frac{n_o^2+n_v^2+n_s^2+n_c^2}{2} V_8 + (n_o n_v + n_s n_c) O_8 - (n_v n_s + n_o n_c) S_8 - (n_v n_c + n_o n_s) C_8$$

$$\mathcal{M}_1 = -\frac{1}{2}(n_o + n_v + n_s + n_c)\hat{V}_8$$

Open descendants of $0B$

- The **open spectra** of these models are **chiral**, and indeed now there are RR tadpoles. Setting them to zero determines **anomaly-free** spectra.

(Polchinski and Cai, 1988)

$$\begin{aligned} NS-NS: &\quad n_o + n_v + n_s + n_c &= 64 \\ R-R: &\quad n_o - n_v - n_s + n_c &= 0 \\ R-R: &\quad n_o - n_v + n_s - n_c &= 0 \end{aligned}$$

- **Gauge groups**: $SO(n_o) \times SO(n_v) \times SO(n_s) \times SO(n_c)$ ($n_o = n_v$, $n_s = n_c$) with **no** dilaton tadpole. Otherwise more possibilities, also all USp's.

- **Technically:** the tadpole conditions remove **irreducible** contributions to anomalies:

$$tr(R^6) \qquad tr(F^6)$$

- Residual **reducible** terms require a (generalized) Green-Schwarz mechanism.

Comments

- A closer look reveals that, for the OB model:

$$\mathcal{A} = \frac{1}{2} \sum_{i,j,k} \mathcal{N}^{i}_{jk} \; \chi_i \; n^j \; n^k$$

- N are **fusion-rule** coefficients for O_8,V_8,$-S_8$,$-C_8$:

$$[\phi_i] \times [\phi_j] = \sum_k \; \mathcal{N}^{k}_{ij} \; [\phi_k]$$

- V_8, rather than O_8, is the **identity**.

- The **Verlinde formula** implies that:

$$\tilde{\mathcal{A}} \sim \sum_{i,j,k,l} \mathcal{N}^{i}_{jk} \; S_{il} \; \chi_l \; n^j \; n^k = \sum_i \chi_i \left(\sum_j \frac{S_{ij} \; n^j}{\sqrt{S_{1j}}} \right)^2$$

- All **perfect squares** in $\tilde{\mathcal{A}}$, as needed to **define** $\tilde{\mathcal{M}}$.

- In $0B$, **one-to-one** correspondence between boundaries and open sectors: **Cardy ansatz**.
(Cardy, 1989)

- In $0A$: **two** types of boundaries, **combinations** of those of $0B$.

Non-tachyonic descendants of $0B$

(A.S., 1995)

- One can modify $\mathcal{K}$ and eliminate the **tachyon** T:

$$\mathcal{K}_3 = \frac{1}{2}(-O_8 + V_8 + S_8 - C_8)$$

$$\tilde{\mathcal{K}}_3 = -\frac{2^6}{2} C_8$$

- **Projected** closed spectrum:

$$e_\mu^a \ , \qquad \phi \ , \qquad B'_{\mu\nu} \ , \qquad \phi'' \ , \qquad D^-_{\mu\nu\rho\sigma}$$

- **Annulus amplitude**: fuse $\mathcal{A}$ for conventional $0B$ descendants with $[-C_8]$ **(complex charges)**:

$$\begin{aligned}\mathcal{A}_3 &= -\frac{n_o^2 + n_v^2 + n_s^2 + n_c^2}{2} C_8 - (n_o n_v + n_s n_c) S_8 \\ &+ (n_v n_s + n_o n_c) O_8 + (n_v n_c + n_o n_s) V_8\end{aligned}$$

- $n_v \to n$, $n_c \to \bar{n}$, $n_o \to m$, $n_s \to \bar{m}$:

$$\begin{aligned}\mathcal{A}_3 &= -\frac{n^2 + \bar{n}^2 + m^2 + \bar{m}^2}{2} C_8 + (n\bar{n} + m\bar{m}) V_8 \\ &+ (\bar{n}m + n\bar{m}) O_8 - (nm + \bar{n}\bar{m}) S_8 \\ \mathcal{M}_3 &= \frac{1}{2}(m + \bar{m} - n - \bar{n}) \hat{C}_8\end{aligned}$$

Comments

- The **closed** sector is **chiral**, and the open sector contains different numbers of L and R fermions. With "complex" charges, the vacuum channel respects **positivity** only because $n = \bar{n}$, $m = \bar{m}$.

$$\tilde{\mathcal{A}}_3 = \frac{2^{-6}}{2}\Big[(n+\bar{n}+m+\bar{m})^2 V_8 - (n+m-\bar{n}-\bar{m})^2 O_8$$

$$- \quad (m-n-\bar{n}+\bar{m})^2 C_8 + (m-n+\bar{n}-\bar{m})^2 S_8\Big]$$

- Here the RR tadpole condition is $m = 32 + n$. $n = 0$: **non-tachyonic spectrum** with gauge group $U(32)$.

- All irreducible anomalies cancel, and the non-abelian irreducible ones are eliminated by a generalized Green-Schwarz mechanism. There is also a $U(1)$ anomaly, and thus the **effective gauge group** is $SU(32)$.

- Several **tachyon-free** compactifications to $D < 10$.

(Angelantonj, 1998)

(Blumenhagen, Font and Lüst, 1999)

(Blumenhagen and Kumar, 1999)

(Förger, 1999)

Toroidal compactification and quantized B_{ab}

(Bianchi, Pradisi and A.S., 1991)

- Begin from the IIB on the 1D torus:

$$\mathcal{T} = |V_8 - S_8|^2 \sum_{m,n \in Z} \frac{q^{\alpha'(m/R+nR/\alpha')^2/4} \bar{q}^{\alpha'(m/R-nR/\alpha')^2/4}}{\eta\bar{\eta}}$$

- Conventional Klein (**with** open sector):

$$\mathcal{K} = \frac{1}{2}(V_8 - S_8)(q^2) \sum_{m \in Z} \frac{q^{\alpha' m^2/2R^2}}{\eta(q^2)}$$

$$\tilde{\mathcal{K}} = \frac{2^5}{2} \frac{R}{\sqrt{\alpha'}} (V_8 - S_8)(i\ell) \sum_{n \in Z} \frac{q^{n^2 R^2/\alpha'}}{\eta(i\ell)}$$

- Modified Klein (**without** open sector):

$$\mathcal{K}' = \frac{1}{2}(V_8 - S_8)(q^2) \sum_{m \in Z} (-1)^m \frac{q^{\alpha' m^2/2R^2}}{\eta(q^2)}$$

$$\tilde{\mathcal{K}}' = \frac{2^5}{2} \frac{R}{\sqrt{\alpha'}} (V_8 - S_8)(i\ell) \sum_{n \in Z} \frac{q^{(n+1/2)^2 R^2/\alpha'}}{\eta(i\ell)}$$

(Dabholkar and Park, 1996)

(Angelantonj et al, 1996)

Open sector and Wilson lines

- Simplest choice does not affect the $SO(32)$ gauge group:

$$\begin{aligned}\mathcal{A} &= \frac{N^2}{2}(V_8 - S_8)(\sqrt{q}) \sum_{m\in Z} \frac{q^{\alpha' m^2/2R^2}}{\eta(\sqrt{q})} \\ \mathcal{M} &= -\frac{N}{2}(\hat{V}_8 - \hat{S}_8)(-\sqrt{q}) \sum_{m\in Z} \frac{q^{\alpha' m^2/2R^2}}{\hat{\eta}(-\sqrt{q})}\end{aligned}$$

- Can be continuously deformed via **Wilson lines**. *e.g.* for SO(32) to U(M) $\times$ SO(N):

$$\begin{aligned}\mathcal{A} &= (V_8 - S_8)(\sqrt{q}) \sum_{m\in Z} \left\{ \left(M\bar{M} + \frac{N^2}{2} \right) \frac{q^{\alpha' m^2/2R^2}}{\eta(\sqrt{q})} \right. \\ &+ MN \frac{q^{\alpha'(m+a)^2/2R^2}}{\eta(\sqrt{q})} + \bar{M}N \frac{q^{\alpha'(m-a)^2/2R^2}}{\eta(\sqrt{q})} \\ &+ \left. \frac{M^2}{2} \frac{q^{\alpha'(m+2a)^2/2R^2}}{\eta(\sqrt{q})} + \frac{\bar{M}^2}{2} \frac{q^{\alpha'(m-2a)^2/2R^2}}{\eta(\sqrt{q})} \right\} \\ \mathcal{M} &= -(\hat{V}_8 - \hat{S}_8)(-\sqrt{q}) \sum_{m\in Z} \left\{ \frac{N}{2} \frac{q^{\alpha' m^2/2R^2}}{\hat{\eta}(-\sqrt{q})} \right. \\ &+ \left. \frac{M}{2} \frac{q^{\alpha'(m+2a)^2/2R^2}}{\hat{\eta}(-\sqrt{q})} + \frac{\bar{M}}{2} \frac{q^{\alpha'(m-2a)^2/2R^2}}{\hat{\eta}(-\sqrt{q})} \right\}\end{aligned}$$

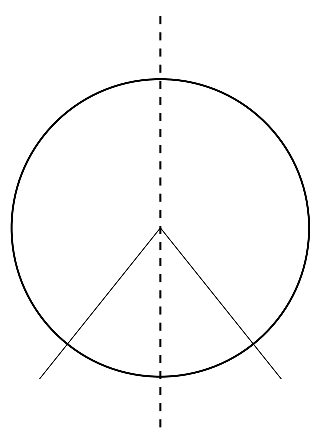
Comments

$$\begin{aligned}\mathcal{A} &= (V_8 - S_8)(\sqrt{q}) \sum_{m \in Z} \left\{ \left(M\bar{M} + \frac{N^2}{2} \right) \frac{q^{\alpha' m^2/2R^2}}{\eta(\sqrt{q})} \right. \\ &+ MN \, \frac{q^{\alpha'(m+a)^2/2R^2}}{\eta(\sqrt{q})} + \bar{M}N \, \frac{q^{\alpha'(m-a)^2/2R^2}}{\eta(\sqrt{q})} \\ &+ \left. \frac{M^2}{2} \, \frac{q^{\alpha'(m+2a)^2/2R^2}}{\eta(\sqrt{q})} + \frac{\bar{M}^2}{2} \, \frac{q^{\alpha'(m-2a)^2/2R^2}}{\eta(\sqrt{q})} \right\}\end{aligned}$$

- For a **integer** $SO(32)$, but already for a **1/2-integer** group **enhancement** to $SO(2m) \times SO(32-2m)$.

- T-duality turns **momentum** translations into **coordinate** translations: **brane displacements**.

(Chaudhuri, Johnson and Polchinski, 1995)



The (quantized) B_{ab}

(Bianchi, Pradisi and A.S., 1991)

- One can consider generalized toroidal compactifications in a **NS-NS** B_{ab} background, as in the Narain heterotic construction:

$$\begin{aligned} p_{L,a} &= m_a + \frac{1}{\alpha'}(g_{ab} - B_{ab})n^b \\ p_{R,a} &= m_a - \frac{1}{\alpha'}(g_{ab} + B_{ab})n^b \end{aligned}$$

- B_{ab} is **not** a mode in the open spectrum: **discrete deformation**.

- **Alternatively**: only a **quantized** B_{ab}, such that $\frac{2}{\alpha'}B_{ab}$ is **integer**, respects the $L \leftrightarrow R$ symmetry.

- The Klein bottle projection is not affected:

$$\begin{aligned} \mathcal{K} &= \frac{1}{2}(V_8 - S_8)(q^2) \sum_m \frac{q^{\frac{\alpha'}{2} m^T g^{-1} m}}{\eta^d(q^2)} \\ \tilde{\mathcal{K}} &= \frac{2^5}{2} \sqrt{\det(g/\alpha')}\ (V_8 - S_8)(i\ell) \sum_n \frac{(e^{-2\pi\ell})^{\frac{1}{\alpha'} n^T g n}}{\eta^d(i\ell)} \end{aligned}$$

The (quantized) B_{ab}

- $\tilde{\mathcal{A}}$ allows only states such that $p_{L,a} = -p_{R,a}$. This introduces a **projector** enforcing the conditions:

$$\frac{2}{\alpha'}\, B_{ab}\, n^b \in Z$$

- As a result, if B_{ab} has an (even) rank r, there are effectively $2^{r/2}$ **images** of the open spectrum that **share** the Chan-Paton charge.

- The **size** of the gauge group is **reduced** by $2^{r/2}$.

$$\tilde{\mathcal{A}} = \frac{2^{r-d-5} N^2}{2} \sqrt{\det(\frac{g}{\alpha'})} (V_8 - S_8) \sum_{n,\epsilon} \frac{\tilde{q}^{\frac{1}{4\alpha'} n^T g n}\, e^{\frac{2i\pi}{\alpha'} n^T B \epsilon}}{\eta^d(i\ell)}$$

$$\mathcal{A} \;=\; \frac{2^{r-d}}{2}\, N^2\, (V_8 - S_8) \sum_{m,\epsilon} \frac{q^{\frac{\alpha'}{2}(m+\frac{1}{\alpha'}B\epsilon)^T g^{-1} (m+\frac{1}{\alpha'}B\epsilon)}}{\eta^d(\sqrt{q})}$$

- The **Möbius** amplitude has some surprising new features, to which we now turn.

The (quantized) B_{ab}

$$\tilde{\mathcal{M}} = -\frac{2N\ 2^{(r-d)/2}}{2}\sqrt{\det(g/\alpha')}(\hat{V}_8 - \hat{S}_8)(i\ell + \tfrac{1}{2}) \times$$

$$\times \sum_{n,\epsilon} \frac{\tilde{q}^{\frac{1}{\alpha'}n^T g n}\, e^{\frac{2i\pi}{\alpha'}n^T B\epsilon}\ \gamma_\epsilon}{\hat{\eta}^d(i\ell + \frac{1}{2})}$$

$$\mathcal{M} = -\frac{2^{(r-d)/2}N}{2}(\hat{V}_8 - \hat{S}_8)\sum_{\epsilon=0,1}\sum_m \frac{q^{\frac{\alpha'}{2}(m+\frac{1}{\alpha'}B\epsilon)^T g^{-1}(m+\frac{1}{\alpha'}B\epsilon)}\ \gamma_\epsilon}{\hat{\eta}^d(-\sqrt{q})}$$

- Here the **projection** is done on **each image**, and introduces the **signs** γ_ϵ.

- The **signs** γ_ϵ ensure the compatibility of the **tadpole condition**, that receives contributions from all values of ϵ, with the direct channel, where the massless terms come only from one value of ϵ.

- Let us look more closely at the simplest case of a **two-torus** with B_{ab}.

The (quantized) B_{ab} and the two-torus

- In this case ($P_{\epsilon_i \epsilon_j}$ is a **shifted** momentum sum):

$$g = \frac{\alpha' Y_2}{X_2} \begin{pmatrix} 1 & X_1 \\ X_1 & |X|^2 \end{pmatrix} \qquad B = \frac{\alpha'}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\tilde{\mathcal{M}} = -\tfrac{2}{2} Y_2 N (\hat{V}_8 - \hat{S}_8)(i\ell + \tfrac{1}{2}) \sum_{n_1,n_2} \frac{W_{n_1,n_2}}{\hat{\eta}^2(i\ell + \frac{1}{2})}$$
$$\times \left[\gamma_{00} + (-1)^{n_1}\gamma_{01} + (-1)^{n_2}\gamma_{10} + (-1)^{n_1+n_2}\gamma_{11}\right]$$
$$\mathcal{M} = -\frac{N}{2}(\hat{V}_8 - \hat{S}_8)\, \frac{[\gamma_{00}P_{00} + \gamma_{01}P_{01} + \gamma_{10}P_{10} + \gamma_{11}P_{11}]}{\hat{\eta}^2(-\sqrt{q})}$$

- From the **tadpole condition**

$$\frac{2^{-5}}{2}(2N)^2 + \frac{2^5}{2} - 2(2N)\left[\frac{1}{2}\sum_{ij} \gamma_{ij}\right] = 0$$

$\sum \gamma_{ij} = 2$: **three** γ are $+1$, **one** is -1.

- **Only** γ_{00} gives massless modes.

- If $\gamma_{00} = 1$ the gauge group is **orthogonal** ($SO(16)$).
  If $\gamma_{00} = -1$ the gauge group is **symplectic** ($USp(16)$).

- Wilson lines **interpolate** between the two.

Orbifold models: T^4/Z_2

- First act with the orbifold on the fermions:

$$V_8 - S_8 = (V_4O_4 - C_4C_4) + (O_4V_4 - S_4S_4) = Q_o + Q_v$$

- The **torus amplitude** is:

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}|Q_o + Q_v|^2[\Lambda_{++}] + \frac{1}{2}|Q_o - Q_v|^2[\Lambda_{+-}] \\ &+ \frac{1}{2}|Q_s + Q_c|^2[\Lambda_{-+}] + \frac{1}{2}|Q_s - Q_c|^2[\Lambda_{--}] \\ Q_s &= O_4C_4 - S_4O_4 \qquad Q_c = V_4S_4 - C_4V_4 \end{aligned}$$

- Working on a generic lattice Λ introduces NN, ND and DD strings.

(Pradisi and A.S., 1989)
(Gimon and Polchinski, 1996)

- Working on a **group lattice** (*e.g.* $SO(8)$) is like in our $D = 10$ examples (rational construction), but there is a surprise: in general **several tensor multiplets**. For comparison, in **heterotic** constructions, **one tensor multiplet**.

(Bianchi and A.S., 1990)

- Group lattices **include** a B_{ab}, and the rank reduction is correlated to the number of tensor multiplets.

The simplest rational models

(Bianchi and A.S., 1990)

- Here we take for Λ_{++} the $SO(8)$ lattice:

$$\begin{aligned} \Lambda_{++} &= |O_4O_4 + V_4V_4|^2 + |V_4O_4 + O_4V_4|^2 \\ &+ |C_4C_4 + S_4S_4|^2 + |S_4C_4 + C_4S_4|^2 \end{aligned}$$

- The whole partition function becomes a **diagonal** modular invariant built out of 16 characters:

$$\mathcal{T} = \sum_{i=1}^{16} |\chi_i|^2$$

whose **identity** is:

$$\chi_1 = Q_o\ O_4O_4 + Q_v\ V_4V_4$$

- χ_1 and **five** other characters contain **massless** states. **Before** Klein projection the latter contain four $(2,0)$ **tensor multiplets** each.

- The Klein projects $[(B_{\mu\nu})_s + (\varphi)_a] \times [3_s + 1_a]$, and one is left with a total of **five** $(1,0)$ tensor multiplets and 16 hypermultiplets.

- How about the Green-Schwarz mechanism here?

The $U(16) \times U(16)$ model (1 tensor)

(Bianchi and A.S., 1990)
(Gimon and Polchinski, 1996)

- Can be obtained introducing a **discrete deformation** in a more complicated rational model, or taking for Λ_{++} a product of **four circles** (**no** B_{ab}).

$$\mathcal{K} = \frac{1}{4}\left[(Q_o + Q_v)(P + W) + 2 \times 16 (Q_s + Q_c)\left(\frac{\eta}{\vartheta_4}\right)^2\right]$$

$$\tilde{\mathcal{K}} = \frac{2^5}{4}\left[(Q_o + Q_v)(vW^e + \frac{P^e}{v}) + 2(Q_o - Q_v)\left(\frac{2\eta}{\vartheta_2}\right)^2\right]$$

- P (W) denote lattice sums **restricted** to zero windings (momenta). P^e (W^e) denote P (W) **restricted** to even momenta (windings).

- Here the terms at the **origin** of the lattice complete "perfect squares" (internal part **implicit**):

$$\tilde{\mathcal{K}}_0 = \frac{2^5}{4}\left[Q_o(\sqrt{v} + \frac{1}{\sqrt{v}})^2 + Q_v(\sqrt{v} - \frac{1}{\sqrt{v}})^2\right]$$

- We can now write the **untwisted terms** in $\tilde{\mathcal{A}}_0$:

$$\tilde{\mathcal{A}}_{0,(unt)} = \frac{2^{-5}}{4}\left[Q_o(N\sqrt{v} + \frac{D}{\sqrt{v}})^2 + Q_v(N\sqrt{v} - \frac{D}{\sqrt{v}})^2\right]$$

The $U(16)\times U(16)$ model

- $\tilde{\mathcal{M}}_0$ is **determined** by $\tilde{\mathcal{K}}_0$ and $\tilde{\mathcal{A}}_0$:

$$\tilde{\mathcal{M}}_0 = -\frac{2}{4}\Big[(Q_o+Q_v)(Nv+\frac{D}{v})+(Q_o-Q_v)(N+D)\Big]$$

- Including the **full** lattice sums:

$$\tilde{\mathcal{A}}_{(unt)} = \frac{2^{-5}}{4}\Big[(Q_o+Q_v)(N^2vW+D^2\frac{P}{v})+2ND(Q_o-Q_v)\Big]$$

$$\tilde{\mathcal{M}} = -\frac{2}{4}\Big[(Q_o+Q_v)(NW^e v+D\frac{P^e}{v})+(Q_o-Q_v)(N+D)\Big]$$

- Reverting to the **direct** channel by S and P, and adding the **breaking terms** in $\mathcal{A}$:

$$\mathcal{A} = \frac{1}{4}\Big[(Q_o+Q_v)(N^2P+D^2W)+2ND(Q_s+Q_c)\Big] + \frac{1}{4}\Big[(Q_o-Q_v)(R_N^2+R_D^2)+2R_NR_D(Q_s-Q_c)\Big]$$

$$\mathcal{M} = -\frac{1}{4}\Big[(Q_o+Q_v)(NP+DW)-(N+D)(Q_o-Q_v)\Big]$$

- Breaking terms in $\tilde{\mathcal{A}}$ display the **brane geometry**:

$$\tilde{\mathcal{A}}_{(tw)} \sim Q_s\left[\left(\frac{R_N}{4}-R_D\right)^2+\frac{15}{16}R_N^2\right]$$

The $U(16) \times U(16)$ model

- **Massless spectrum**: gauge group $U_N \times U_D$, since the gauge vectors (Q_o) do not appear in $\mathcal{M}_0$.

$$\mathcal{A}_0 = \frac{1}{4}\Big[(Q_o{+}Q_v)(N^2 + D^2){+}2NDQ_s\Big] + \frac{1}{4}\Big[(Q_o{-}Q_v)(R_N^2 + R_D^2){+}2R_N R_D Q_s\Big]$$

$$\mathcal{M}_0 = -\frac{1}{2}Q_v(N + D)$$

- With "complex" charges:

$$N{=}n + \bar{n} \quad R_n{=}i(n - \bar{n}) \quad D{=}d + \bar{d} \quad R_d{=}i(d - \bar{d})$$

$$\mathcal{A}_0 + \mathcal{M}_0 = (n\bar{n} + d\bar{d})Q_o{+}(nd + \bar{n}\bar{d})Q_s + \frac{n(n-1) + \bar{n}(\bar{n}-1) + d(d-1) + \bar{d}(\bar{d}-1)}{2}Q_v$$

- **Tadpole conditions:** give $U(16) \times U(16)$, one of the cases discussed with Bianchi (1990). Method as in the paper with Pradisi (1989).
  However: can **deform** by **continuous** Wilson lines / brane displacements, adding images. I was confused on this point, clarified by Gimon and Polchinski.

Other SUSY models

- **6D** orbifold models with variable numbers of tensor multiplets (**including zero**!).

  (Bianchi and A.S., 1990;Dabholkar and Park;Gimon and Johnson;Angelantonj et al;Ibanez et al;Blumenhagen et al,..)

- **4D chiral** orbifold models (simplest: descendants of T^6/Z_3). **Perturbative** heterotic duals.

  (Angelantonj et al, 1996;Kakushadze et al;Ibanez et al; Zwart;..)

- **Genuinely curved** models.

  *(Bianchi and Stanev, 1997;***Blumenhagen and Wißkirchen***; Stanev et al)*

- Relation with rational models: B_{ab} orders the fixed points in multiplets, and allows $U \leftrightarrow USp$ transitions. In 6D models, it alters the number of tensor multiplets.

  (Kakushadze, Shiu, Tye, 1998; **Angelantonj***, 1999)*

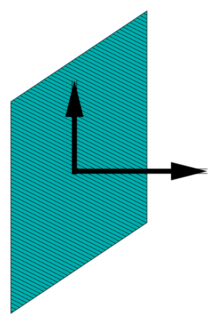
Models with broken supersymmetry

- Begin with the **Scherk-Schwarz mechanism** (SS, 1979): deform Kaluza-Klein reductions using a **global symmetry**. If bosons and fermions are treated differently: **SUSY breaking**.

- Field Theory: **continuous** deformations. (Closed) String Theory: **discrete** deformations: $M_s \sim 1/R$.

- In **open-string** models: one more datum, the **spatial relation** between SS deformations and the vacuum brane configuration.

(Ferrara, Kounnas, Porrati, Zwirner, 1988)



- **Parallel**: conventional (open and closed) SS.
  Orthogonal: brane supersymmetry.

(Antoniadis,Dudas, A.S., 1998; Antoniadis,D'Appollonio,Dudas, A.S.;Kakushadze and Tye; Blumenhagen and Görlich; Angelantonj,Antoniadis, Förger)

Brane supersymmetry breaking

(Antoniadis, Dudas, A.S., 1999)

- In some models, it is apparently **impossible** to solve some of the tadpole conditions.
- A solution can actually be found introducing boundaries that respect only part of the bulk symmetry.
- **In particular:** one can construct models where tadpole cancellation requires that the vacuum configuration involve both **branes** and **anti-branes**, in stable configurations, so that supersymmetry is **necessarily** broken at the string scale on some branes, there are no tachyons, but lowest order the **bulk** is **supersymmetric**.
- There are several classes of models that **require** this type of solution. Moreover, one can extend the vacuum configurations to allow stable configuration of **branes** and **anti-branes**, with net numbers fixed by the tadpole conditions.

  (Aldazabal, Uranga, Ibanez, 1999)
  (Angelantonj, Antoniadis, D'Appollonio, Dudas, A.S., 1999)

Brane supersymmetry breaking: an example

- Consider again the T^4/Z_2 orbifold, and change the Klein-bottle projection to:

$$\mathcal{K} = \frac{1}{4}\left[(Q_o + Q_v)(P+W) - 2 \times 16 (Q_s + Q_c)\left(\frac{\eta}{\vartheta_4}\right)^2\right]$$

$$\tilde{\mathcal{K}}_0 = \frac{2^5}{4}\left[Q_o(\sqrt{v} - \frac{1}{\sqrt{v}})^2 + Q_v(\sqrt{v} + \frac{1}{\sqrt{v}})^2\right]$$

- Now the projection leaves 17 **tensors**.

- **Naively:** change $D \to -D$ in the previous equations, but this is incompatible with positivity.

- **Solution**: introduce anti-branes, breaking supersymmetry at the string scale. Chiral, anomaly free model.

$$Q'_o = V_4 O_4 - S_4 S_4 \quad Q'_v = O_4 V_4 - C_4 C_4$$

$$\tilde{\mathcal{A}}_{(unt)} = \frac{2^{-5}}{4}\left[(Q_o + Q_v)(N^2 v W + D^2 \frac{P}{v}) + 2ND(Q'_o - Q'_v)\right]$$

- The RR ND contribution has been **inverted**. All RR tadpoles can be solved, while the $NS-NS$ ones give a contribution to the **vacuum energy** localized on the anti-branes.

3. Lessons from and for Low-Energy Supergravity

a. The generalized Green-Schwarz mechanism.

b. Low-energy tensor-vector couplings.

c. Covariant and consistent forms.

d. Lack of uniqueness and central extension.

e. Some subtleties.

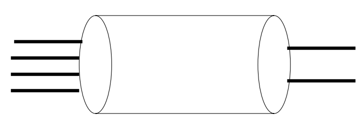
The generalized Green-Schwarz mechanism

- As we have seen, $6D$ open-string models typically involve several antisymmetric tensors, left over by the Klein-bottle projection.

- In $D = 10$, the single antisymmetric tensor present in type-I and heterotic models cancels via **non gauge invariant** contact interactions some **reducible** anomalies.

(Green and Schwarz, 1984)

- The corresponding diagrams in String Theory are free of anomalies, since they are **regulated** by the momentum flow.



- The contact interactions, summarized in the anomaly polynomial, are already **more general** for the descendants of $0B$ models.

(A.S., Buckow '96)

The generalized Green-Schwarz mechanism

(A.S., 1992)

- **However**: $10D$ Green-Schwarz couplings are **higher-derivative**, **not** in the low-energy field theory.

- **On the other hand**: **some** $6D$ Green-Schwarz couplings, related to gauge anomalies, are in the low-energy field theory.

- **Unconventional** supergravity couplings:

1. Classical field equations with **anomalies**.

2. **Extensions** of the supersymmetry algebra.

3. Non-unique Lagrangian: WZ consistency conditions.

4. **Singularities** related to a **new type of phase transition**: strings becoming **tensionless**.

(Duff, Minasian and Witten, 1996)

(Duff, Lü and Pope; Seiberg and Witten, 1996)

Some Six - Dimensional Multiplets

- **(1,0) supersymmetry:**

  gravity multiplet : $\left(e_\mu^m, \psi^a_{\mu,L}, B^+_{\mu\nu}\right)$

  tensor multiplet : $\left(B^-_{\mu\nu}, \chi^a_R, \phi\right)$

  vector multiplet : $\left(A_\mu, \lambda^a_L\right)$

  hypermultiplet : $\left(4\phi, \psi^a_R\right)$

- **(2,0) supersymmetry:**

  gravity multiplet: $\left(e_\mu^m, 2\psi^a_{\mu,L}, 5B^+_{\mu\nu}\right)$

  tensor multiplet: $\left(B^-_{\mu\nu}, 2\chi^a_R, 5\phi\right)$

  All spinors are **symplectic** Majorana-Weyl:

$$\psi^a = \pm\gamma_7\psi^a \qquad \psi^a = \Omega^{ab} C \bar{\psi}_b$$

- **(2,0) supergravity : unique** anomaly-free combination of gravity multiplet and 21 tensor multiplets. The scalars parametrize $SO(5,21)/SO(5)\times SO(21)$. The field equations, originally constructed to lowest order in the Fermi fields, have been recently completed to all orders.

(Romans, 1986)
(Riccioni, 1997)

(1,0) supergravity + tensor multiplets

- Here we confine our attention to the field equations.

- Romans (1996): the scalars parameterize the **coset** $SO(1,n_T)/SO(n_T)$, described by the matrix

$$V^T = (v_r \quad x_r^m)$$

whose elements satisfy the constraints:

$$v^r v_r = 1 \qquad v^r x_r^m = 0 \qquad v_r v_s - x_r^m x_s^m = \eta_{rs}$$

- Defining $G_{rs} = v_r v_s + x_r^m x_s^m$, the conditions

$$G_{rs} H^{s\mu\nu\rho} = \frac{1}{6e} \epsilon^{\mu\nu\rho\alpha\beta\gamma} H_{r\alpha\beta\gamma}$$

imply that $v_r H^r_{\mu\nu\rho}$ is **self-dual**, while the n_T tensors $x_r^m H^r_{\mu\nu\rho}$ are **antiself-dual**.

- The divergence yields **second-order** equations:

$$D_\mu (G_{rs} H^{s\mu\nu\rho}) = 0$$

(1,0) supergravity + tensor multiplets

- This is a **conventional** supergravity construction.
- One begins from the bosonic couplings in the bosonic equations and from the terms linear in the fermions in the fermionic equations.
- The **lowest-order** fermionic equations can be turned into their bosonic partners, using the **lowest-order** supersymmetry transformations, at most linear in the fermions:

$$
\begin{aligned}
\delta e_\mu{}^a &= -i(\bar{\epsilon}\gamma^a\Psi_\mu) \\
\delta B^r_{\mu\nu} &= i v^r (\bar{\Psi}_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2} x^{mr} (\bar{\chi}^m \gamma_{\mu\nu}\epsilon) \\
\delta v_r &= x^m_r (\bar{\epsilon}\chi^m) \\
\delta \Psi_\mu &= D_\mu \epsilon + \frac{1}{4} v_r H^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon \\
\delta \chi^m &= \frac{i}{2} x^m_r \partial_\mu v^r \gamma^\mu \epsilon + \frac{i}{12} x^m_r H^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon
\end{aligned}
$$

(1,0) supergravity + tensor multiplets

- The lowest-order **equations** are :

$$
\begin{aligned}
\gamma^{\mu\nu\rho} D_\nu \Psi_\rho &+ v_r H^{r\mu\nu\rho}\gamma_\nu \Psi_\rho - \frac{i}{2} x_r^m H^{r\mu\nu\rho}\gamma_{\nu\rho}\chi^m \\
&+ \frac{i}{2} x_r^m \partial_\nu v^r \gamma^\nu \gamma^\mu \chi^m = 0 \\
g^\mu D_\mu \chi^m &- \frac{1}{12} v_r H^{r\mu\nu\rho}\gamma_{\mu\nu\rho}\chi^m - \frac{i}{2} x_r^m H^{r\mu\nu\rho}\gamma_{\mu\nu}\Psi_\rho \\
&- \frac{i}{2} x_r^m \partial_\nu v^r \gamma^\mu \gamma^\nu \Psi_\mu = 0 \\
x_r^m D_\mu(\partial^\mu v^r) &+ \frac{2}{3} x_r^m v_s H^r_{\alpha\beta\gamma} H^{s\alpha\beta\gamma} = 0 \\
R_{\mu\nu} &- \frac{1}{2} g_{\mu\nu} R + \partial_\mu v^r \partial_\nu v_r - \frac{1}{2} g_{\mu\nu}\partial_\alpha v^r \partial^\alpha v_r \\
&- G_{rs} H^r_{\mu\alpha\beta} H^s{}_\nu{}^{\alpha\beta} = 0
\end{aligned}
$$

- The lowest-order **gauge algebra** is :

$$
\begin{aligned}
[\delta_1, \delta_2] &= \delta_{gct}(\xi^\mu = -i(\bar{\epsilon}_1 \gamma^\mu \epsilon_2)) \\
&+ \delta_{tens}(\Lambda^r_\mu = -\frac{1}{2} v^r \xi_\mu - \xi^\nu B^r_{\mu\nu}) \\
&+ \delta_{SO(n)}(A^{mn} = \xi^\mu x^{mr}(\partial_\mu x^n_r)) \\
&+ \delta_{Lorentz}(\Omega^{ab} = -\xi_\mu(\omega^{\mu ab} - v_r H^{r\mu ab}))
\end{aligned}
$$

(1,0) supergravity + tensor multiplets

- One can complete these equations following an (almost) standard, albeit tedious, procedure.
  Basic steps are :

1. The supercovariantization of fermionic transformations.

2. The requirement that the gauge algebra close "on shell".

3. The addition of higher-order fermionic terms to the spin connection and to the transformation law of the gravitino.

- **Tool**: Fierz identities for $Sp(2)$ doublets.

4. The use of the complete gauge algebra to recover the fermionic equations.

(Schwarz, 1983)

5. The complete fermionic equations are :

$$\gamma^\mu \hat{D}_\mu \chi^m - \frac{1}{12} v_r \hat{H}^r_{\mu\nu\rho} \gamma^{\mu\nu\rho} \chi^m - \frac{i}{2} x^m_r \hat{H}^{r\mu\nu\rho} \gamma_{\mu\nu} \Psi_\rho$$
$$-\frac{i}{2} x^m_r (\widehat{\partial_\nu v^r}) \gamma^\mu \gamma^\nu \Psi_\mu - \frac{i}{2} \gamma^\alpha \chi^n (\bar{\chi}^n \gamma_\alpha \chi^m) = 0$$

$$\gamma^{\mu\nu\rho} \hat{D}_\nu \Psi_\rho + \frac{1}{4} v_r \hat{H}^r_{\nu\alpha\beta} \gamma^{\mu\nu\rho} \gamma^{\alpha\beta} \Psi_\rho - \frac{i}{2} x^m_r \hat{H}^{r\mu\nu\rho} \gamma_{\nu\rho} \chi^m$$
$$+ \frac{i}{2} x^m_r (\widehat{\partial_\nu v^r}) \gamma^\nu \gamma^\mu \chi^m + \frac{3i}{2} \gamma^{\mu\alpha} \chi^m (\bar{\chi}^m \Psi_\alpha)$$
$$- \frac{i}{4} \gamma^{\mu\alpha} \chi^m (\bar{\chi}^m \gamma_{\alpha\beta} \Psi^\beta) + \frac{i}{4} \gamma_{\alpha\beta} \chi^m (\bar{\chi}^m \gamma^{\mu\alpha} \Psi^\beta)$$
$$- \frac{i}{2} \chi^m (\bar{\chi}^m \gamma^{\mu\alpha} \Psi_\alpha) = 0$$

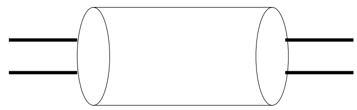
6. Finally, **deduce** the complete bosonic equations from a Lagrangian that gives them (aside from the tensor equation), and prove local supersymmetry :

$$\delta F \frac{\delta \mathcal{L}}{\delta F} + \delta B \frac{\delta \mathcal{L}}{\delta B} = 0$$

where F and B denote collectively the Fermi and Bose fields aside from the antisymmetric tensors.

Tensor - vector couplings

(A.S., 1992; Ferrara, Minasian and A.S.; Nishino and Sezgin;Ferrara, Riccioni and A.S.)



- Unfamiliar features, introduced by Green-Schwarz couplings related to the **residual** anomaly polynomial. In open-string vacua, after **tadpole conditions**:

$$I_8 = -\sum_{x,y} \; c_x^r \; c_y^s \; \eta_{rs} \; \text{tr}_x F^2 \; \text{tr}_y F^2$$

1. Include the Chern-Simons couplings

$$H^r = dB^r - c^{rz}\omega_z$$

- Now B^r must transform under vector gauge transformations according to

$$\delta B^r = c^{rz} tr_z(\Lambda dA)$$

- The divergence of the (anti)self-duality conditions now involves the "instanton density":

$$D_\mu(G_{rs} H^{s\mu\nu\rho}) = -\frac{1}{4e}\epsilon^{\nu\rho\alpha\beta\gamma\delta} c_r^z tr_z(F_{\alpha\beta}F_{\gamma\delta})$$

2. The field equations now include some additional couplings, obtained requiring **supersymmetry** to lowest order. In particular, the anomaly couplings determine the **vector equation**:

$$D_\mu(v_r c^{rz} F^{\mu\nu}) - c^{rz} G_{rs} H^{s\nu\rho\sigma} F_{\rho\sigma} = 0$$

- **Moduli-dependent gauge couplings:**

$$\frac{1}{g^2_{eff,z}} \sim v^r c^r_z$$

that effectively limit the moduli to their **region of positivity**. Actually **inconsistent**, since the effective gauge current is **not** divergence-free:

$$D_\mu J^\mu = -\frac{1}{2e} \epsilon^{\mu\nu\alpha\beta\gamma\delta} c^{rz} c^{z'}_r F_{\mu\nu} tr_{z'}(F_{\alpha\beta} F_{\gamma\delta})$$

- A "classical" field theory with **anomalies**, and a number of subtle properties usually ascribed to fermion determinants! Here we see the **covariant anomaly**, since the construction was based on the requirements of **covariance** and **local supersymmetry**.

- We can also revert to the **consistent anomaly**, trading the condition of local supersymmetry for the solution of **Wess-Zumino** consistency conditions.

- Already to lowest order

$$\delta_\Lambda \mathcal{A}_\epsilon = \delta_\epsilon \mathcal{A}_\Lambda$$

implies the presence of a **supersymmetry anomaly**:

$$\begin{aligned}\mathcal{A}_\epsilon = & -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu A_\nu) tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \\ & -\frac{1}{6}\epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{rz'} tr_z(\delta_\epsilon A_\mu F_{\nu\alpha})\omega^{z'}_{\beta\gamma\delta}\end{aligned}$$

- This modifies the vector equation, that is now **integrable** :

$$\begin{aligned}& D_\mu(v_r c^{rz} F^{\mu\nu}) - \frac{1}{8e}\epsilon^{\nu\rho\alpha\beta\gamma\delta} c_r^z A_\rho c^{rz'} tr_{z'}(F_{\alpha\beta}F_{\gamma\delta}) \\ - \; & G_{rs} H^{s\nu\rho\sigma} c^{rz} F_{\rho\sigma} - \frac{1}{12e}\epsilon^{\nu\rho\alpha\beta\gamma\delta} c_r^z F_{\rho\alpha} c^{rz'} \omega^{z'}_{\beta\gamma\delta} = 0\end{aligned}$$

- The divergence of the gauge current is now the **consistent anomaly**:

$$\mathcal{A}_\Lambda = -\frac{1}{4}\epsilon^{\mu\nu\alpha\beta\gamma\delta} c_r^z c^{rz'} tr_z(\Lambda\partial_\mu A_\nu) tr_{z'}(F_{\alpha\beta}F_{\gamma\delta})$$

Tensor - vector couplings

- Now one can try to proceed as in standard supergravity constructions, fixing the **WZ consistency conditions** with higher-order fermionic terms.

4. The supersymmetry transformation of the gravitino is modified, and the spin connection contains an additional term depending on the gauginos.

5. As anticipated by Nishino and Sezgin (1997), the complete fermionic transformations and couplings contain singular terms, proportional to $g^2_{eff,z}$.

6. Again, the procedure determines the field equations of χ^m and ψ_μ and the supersymmetry transformations.

Tensor - vector couplings

7. One can finally derive the bosonic equations and show that:

$$\delta B \frac{\delta \mathcal{L}}{\delta B} + \delta F \frac{\delta \mathcal{L}}{\delta F} = \mathcal{A}_\epsilon$$

However:

8. The result is not unique!

9. The commutator on the gauginos **does not close**, not even on the field equations. Rather, it displays the presence of an **extension**.

- **Actually:** the extension guarantees the **consistency** of the (non-unique) construction !

One more surprise

(Riccioni and A.S., 1998)

- We know that the divergence of the **vector equation** gives the consistent gauge anomaly

$$D_\mu J^\mu_\Lambda = -\mathcal{A}_\Lambda$$

- The divergence of the Rarita-Schwinger equation gives the consistent supersymmetry anomaly

$$D_\mu J^\mu_\epsilon = -\mathcal{A}_\epsilon$$

- How about the divergence of the **Einstein equation**? Since we are not taking into account the gravitational part of the residual anomaly, should we expect to find $\nabla_\nu T^{\mu\nu} = 0$?

- **No !** And, indeed, the Einstein equation is also (correctly) **inconsistent**. It "feels" the inconsistency of the other equations, since **all fields** have derivative transformations under *g.c.t.* !

$$\delta_\xi \mathcal{L} = -2\xi_\mu \nabla_\nu T^{\mu\nu} - \xi_\nu tr_z(A^\nu \nabla_\mu J^\mu) - \xi_\nu(\bar{\Psi}^\nu \nabla_\mu \tilde{J}^\mu)$$

4. Lessons from and for Boundary CFT

a. Bulk and boundary/crosscap CFT.

b. Sewing constraints.

c. Examples.

d. Polynomial equations for $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$.

Bulk CFT and (closed) String Theory

- After the 1984 paper of *Belavin, Polyakov and Zamolodchikov*, **bulk 2D CFT** was soon recognized as the **key ingredient** of String Theory:

1. it provided the tools for (tree) superstring amplitudes;

   (Friedan, Martinec and Shenker, 1985)

2. it helped to clarify the role of **modular invariance**;

   (Cardy, 1986)

3. it made systematic studies of **orbifold-like** closed string vacua possible;

   (Dixon and Harvey; Seiberg and Witten; Dixon, Harvey, Vafa and Witten, 1985; Narain, Sarmadi and Vafa; Antoniadis, Bachas and Kounnas; Kawai, Lewellen and Tye; Lerche, Lüst and Schellekens, 1987)

4. it led to the elegant Gepner construction of **genuinely curved** closed string vacua.

   (Gepner, 1987)

... and more ...

- However, **bulk 2D CFT** soon developed into a large field, and underwent important structural developments.

- **Basic idea**: (hidden) analytic structure reduces 2D CFT's to (combinations of) one-dimensional systems.

$$f(z_i, \bar{z}_i) = \sum_{k,k'=1}^{n} c_{k,k'}\, f_k(z_i)\, \bar{g}'_k(\bar{z}_i)$$

- **Soluble** models when n is (somehow) **finite**, *i.e.* when the conformal symmetry (or some extension) arranges the operators into a **finite** number of families:

Rational CFT (RCFT)

- A major effort to classify **RCFT's** was only partly successful, but led to the elegant ADE **classification** of $sl(2)$ and minimal modular invariants.

(Cappelli, Itzykson, Zuber, 1987)

- Within RCFT one can address problems more precisely. The **torus amplitude** encodes the **operator content** and is severely constrained by **modular invariance**. Let us write it in the form:

$$\mathcal{T} = \sum_{i,j=1}^{n} \bar{\chi}_i \, X_{ij} \, \chi_j$$

- Let us also assume that **each pair** (i,j), and in fact **each chiral label**, identifies an **operator family** $[\phi_i]$. Powerful techniques have been developed to resolve **ambiguities**.

(Schellekens and Yankielowicz, 1989)

(Fuchs, Schellekens and Schweigert, 1996)

- The action of T:$\tau \to \tau+1$ and S:$\tau \to -1/\tau$ on the n **characters** is then described by **unitary** matrices, T and S, with T **diagonal** and S **symmetric**, such that:

$$S^2 = (ST)^3 = \mathcal{C}$$

- $\mathcal{C}$ is called the **conjugation** matrix of the RCFT.

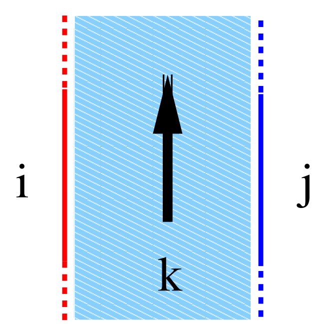
- The **Verlinde formula** relates the **fusion-rule coefficients** $\mathcal{N}^{i}_{jk}$ to the S matrix:

$$[\phi_i] \times [\phi_j] = \sum_k \mathcal{N}^{k}_{ij} \, [\phi_k]$$

$$\mathcal{N}^{i}_{jk} = \sum_l \frac{S_{jl} \, S_{kl}}{S_{1l}} \, {S^\dagger}_{il}$$

- This expression depends on **chiral** data: can one link it to the **annulus** amplitude? For the **charge-conjugation** modular invariant ($X = \mathcal{C}$): **one-to-one** correspondence between (open) **sectors** and types of **boundaries**.

(Cardy, 1989)



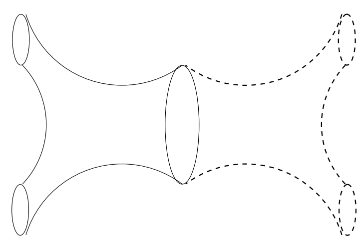
- **Operator content** for boundaries i and j:

$$\mathcal{A}_{ij} = \sum_k \mathcal{N}^{k}_{ij} \, \chi_k$$

Bulk CFT: sewing constraints

- We have seen that 2D CFT amplitudes have **analytic** conformal blocks $f_k(z_i)$. If we **know** the conformal blocks (or their generalizations), how should we **assemble** them?

sewing constraints



- These **global** constraints:

1. guarantee the equivalence of different decompositions of amplitudes into **building blocks** (here 3-point vertices for **bulk operators**).

2. Equivalently, they guarantee the **equivalence** of different Laurent expansions for the amplitudes.

Sewing constraints

- One can specify a 2D CFT in terms of **basic data**:

1. **central charge** c;

2. **conformal weights** $(h_i, \bar{h}_i)$, with $(h_i - \bar{h}_i) \in Z$;

3. **OPE coefficients** ${C_{i\bar{\imath}j\bar{\jmath}}}^{k\bar{k}}$:

$$\phi_{i,\bar{\imath}}(z,\bar{z})\,\phi_{j,\bar{\jmath}}(w,\bar{w}) \sim \sum_{k,\bar{k}} {C_{i\bar{\imath}j\bar{\jmath}}}^{k\bar{k}}\,(z-w)^{h_i+h_j-h_k} \times$$
$$\times\ (\bar{z}-\bar{w})^{\bar{h}_i+\bar{h}_j-\bar{h}_k}\,\phi_{k,\bar{k}}(w,\bar{w})$$

- **In principle**: the $C's$ are **three-point** couplings, that allow one to build arbitrary string amplitudes by **sewing**. What is the **minimal** number of **constraints** for a **unique sewing procedure**?

1. four-point function on the **sphere**.

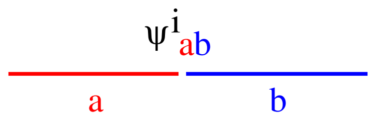
2. one-point function on the **torus**.

(Sonoda, 1988)

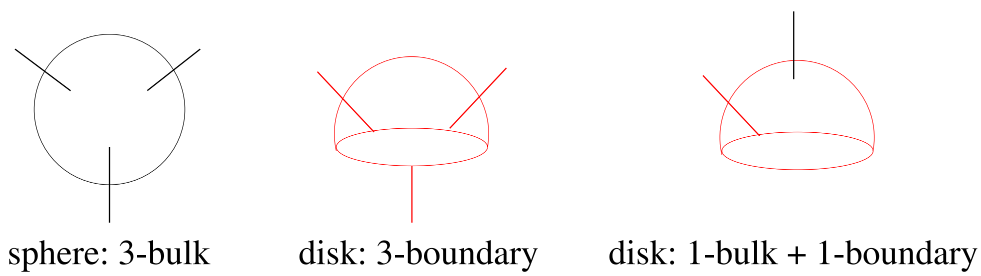
Bulk and boundary CFT

- If boundaries are present, there are both **bulk** (closed-string) and **boundary** (open-string) operators.

- In general, a **boundary operator** ψ_i^{ab} **changes** the boundary condition (here along the real axis):

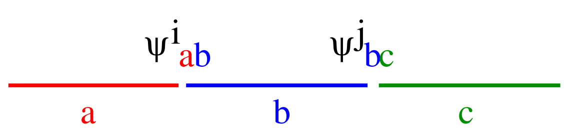


- Now the **building blocks** are:



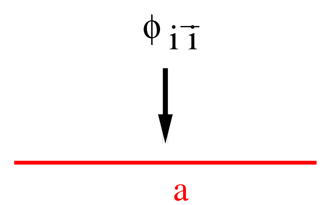
- The second rests on the **boundary** OPE ($x < y$):

$$\psi^i_{ab}(x)\,\psi^j_{bc}(y) \sim \sum_k C^{abc}_{ijk}\,(y-x)^{h_i+h_j-h_k}\,\psi^k_{ac}(y)$$



- The third rests on the **bulk-to-boundary** OPE :

$$\phi_{i,\bar{\imath}}(z,\bar{z}) \sim \sum_k C^a_{(i\bar{\imath})k}\,(z-\bar{z})^{h_i+h_{\bar{\imath}}-h_k}\,\psi^k_{aa}(Re(z))$$



- **New** constraints with **boundaries/crosscaps**?

1. The **annulus** constraint.
(Lovelace, 1971; Cardy, 1989)

2. The **planar duality** for boundary 4-points.
(Veneziano, 1968)

3. The **2 boundary - 1 bulk** constraint.

4. The **1 boundary - 2 bulk** constraint.
(Cardy and Lewellen; Lewellen, 1991)

5. The **crosscap** constraint.
(Fioravanti, Pradisi and A.S., 1993; Pradisi, A.S. and Stanev, 1995)

CFT with boundaries and crosscaps

- The basic data are:

1. c : **central charge**;

2. $(h_i \bar{h}_i)$: **conformal weights** of **bulk** operators;

3. $\tilde{h}_i$: **conformal weights** of **boundary** operators;

4. $C_{i\bar{\imath}j\bar{\jmath}}{}^{k\bar{k}}$: **bulk** OPE coefficients;

5. C_{ijk}^{abc} : **boundary** OPE coefficients;

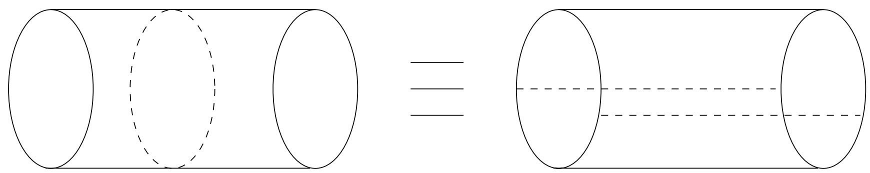
6. $C_{(i\bar{\imath})k}^{a}$: **bulk-to-boundary** OPE coefficients. In particular, if k is the identity they are the **boundary** one-point functions (**tadpole** coefficients) B_i^a;

7. Γ_i : **crosscap** one-point functions, or **tadpole** coefficients.

- **Sewing constraints**: (non)linear relations.

The annulus constraint

1. The **annulus constraint** is:



- The two (**direct** and **transverse**) forms for the annulus are related by an S transformation:

$$\mathcal{A} \;=\; \sum_{\alpha\beta i} n^{\alpha}\, n^{\beta}\, \mathcal{A}^{i}_{\alpha\beta}\;\chi_i$$
$$\tilde{\mathcal{A}} \;=\; \sum_{i} \chi_i \left(B^{i}_{\alpha}\, n^{\alpha}\right)^2$$

- Compatibility implies the **spectral decompositions**:

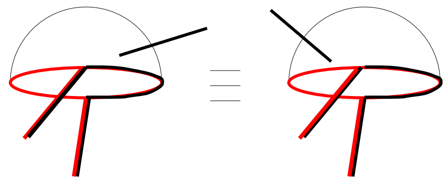
$$\mathcal{A}^{i}_{\alpha\beta} = \sum_{j} B^{j}_{\alpha}\, B^{j}_{\beta}\, S^{\dagger}_{ij}$$

- **Range of j**: as in the 10D, only bulk families **allowed** in tube.

- For the $\mathcal{C}$-conjugate case, $B^{i}_{\alpha} \to S_{ij}/\sqrt{S_{1j}}$, one recovers the **Verlinde formula**.

(Cardy, 1989)

3. The **2 boundary - 1 bulk constraint** is:



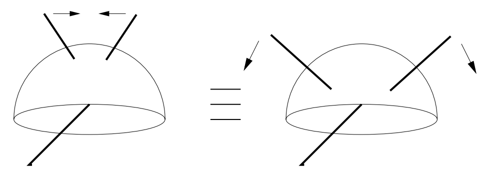
- It enforces the **consistency** of the limiting behaviors for the bulk operator approaching the two portions of the boundary, and can be written in the form:

$$C^{b}_{(i,\bar{i})l}\, C^{bab}_{jkl}\, {\alpha_l}^{bb} = \sum_{m,n,p} C^{a}_{(i,\bar{i})n} C^{aba}_{kjn}\, {\alpha_n}^{aa} (-1)^{(I_i - I_{\bar{i}} + 2I_j + I_p - I_m)}$$
$$e^{-i\pi(\Delta_i - \Delta_{\bar{i}} - \Delta_m + \Delta_p)} F_{nm}(j,i,\bar{i},k) F^{-1}_{mp}(i,j,\bar{i},k) F_{pl}(i,\bar{i},j,k)$$

- Like all other sewing constraints, this involves the **duality matrices** F of the CFT, that are only known in special cases. Together with the **planar duality** for boundary 4-point functions, it can determine the C^{abc}_{ijk}, once one knows the $C^{a}_{(i\bar{i})l}$.

The 1 boundary - 2 bulk constraint

4. The **1 boundary - 2 bulk constraint** is:



- It enforces the **consistency** of two limiting behaviors, for the two bulk operators next to each other of next to the boundary.

- **Quadratic relations** for the $C^a_{(i\bar{i})k}$. One subset, corresponding to **empty** boundaries ($k = 1$), defines a **quadratic** system for the B^a_i: in general, several choices for boundary conditions since the quadratic system has **multiple** solutions!

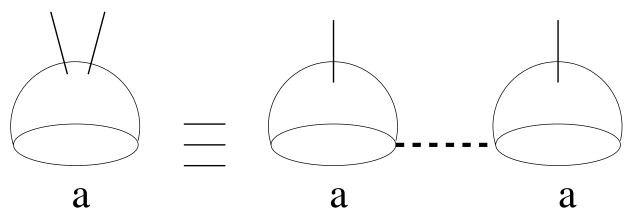
$$B^a_i\, B^a_j = \sum_k {X_{ij}}^k\, B^a_k$$

In D_{odd} $sl(2)$ WZW models, where the ${X_{ij}}^k$ are known, a new phenomenon: **extension** of the boundary symmetry.

(Pradisi, A.S. and Stanev, 1996)

The 2 boundary - 1 bulk constraint

$$B_i^a \, B_j^a = \sum_k X_{ij}{}^k \, B_k^a$$



- These relations may be justified considering the figure, and confining the attention to the exchange of the **identity**.

- They **identify** the B_i^a as one-dimensional irreps of an abelian algebra (**classifying algebra**). Basic problem is obtaining the **structure constants**, but this can be done in several interesting cases.

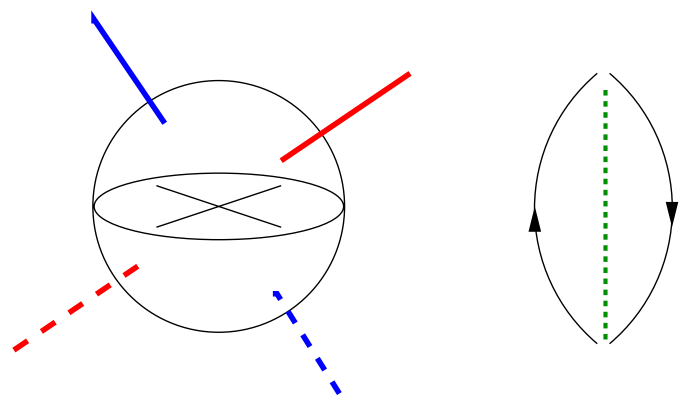
  (Fuchs and Schweigert, 1997)

- At the end we shall see another formulation of the problem, in terms of (positive) **integer matrices**.

The crosscap constraint

(Fioravanti, Pradisi and A.S., 1993)

- This is a set of **linear** equations for the crosscap 1-point functions Γ_i that enforce the **consistency** of two limiting behaviors, when the two operators approach **each other** or their **images**:



- For the Γ_i, it also rests on largely unknown properties of the CFT (bulk structure constants and duality matrices).

- **However**: once the Γ_i are extracted from $\mathcal{K}$, **efficient** way to derive the **bulk structure constants**.

- **Important generalization**: the bulk fields can behave as Z_2 sections (**multiple** choices for $\mathcal{K}$).

(Pradisi, A.S. and Stanev, 1995)

$SU(2)$ WZW models (χ_{2I+1}: isospin I)

- Simplest non-trivial case is A_3:

$$\mathcal{T} = |\chi_1|^2 \; + \; |\chi_2|^2 \; + \; |\chi_3|^2$$

- Since the model is **diagonal**, try **Cardy ansatz**.

$$\mathcal{A}=\left(\frac{n_1^2+n_2^2+n_3^2}{2}\right)\chi_1+n_2(n_1+n_3)\chi_2+\left(\frac{n_2^2}{2}+n_1n_3\right)\chi_3$$

- **However**: here the corresponding $\mathcal{K}$ is:

$$\mathcal{K} \; = \; \frac{1}{2}\left(\chi_1 - \chi_2 + \chi_3\right)$$

- The corresponding $\mathcal{M}$ is:

$$\mathcal{M} \; = \; \pm \left[\; \frac{n_1 - n_2 + n_3}{2} \; \hat{\chi}_1 \; + \; \frac{n_2}{2} \; \hat{\chi}_3 \; \right]$$

- **Ordinary** $\mathcal{K}$ (all symmetrized)? **Complex** charges.

$$\mathcal{A} \; = \; \left(\frac{n_2^2}{2}+m\bar{m}\right)\chi_1+n_2\left(m+\bar{m}\right)\chi_2+\frac{n_2^2+m^2+\bar{m}^2}{2}\chi_3$$

$$\mathcal{M} \; = \; \pm \left[\; \frac{n_2}{2} \; \hat{\chi}_1 \; + \; \frac{n_2 + m + \bar{m}}{2} \; \hat{\chi}_3 \; \right]$$

The A series

1. $\mathcal{K}$ with **alternating** signs: **real** charges.

$$\mathcal{A} = \frac{1}{2} \sum_{a,b,c} N_{ab}^{c} \; n^a \; n^b \; \chi_c$$

$$\mathcal{M} = \pm\frac{1}{2}\sum_{a,b}(-1)^{b-1}(-1)^{\frac{a-1}{2}} N_{bb}^{a}\, n^b\, \hat{\chi}_a \left[= \sum_{ab} n^b\, \hat{\chi}_a\, \mathcal{Y}_{1b}^{a}\right]$$

2. $\mathcal{K}$ with only **positive** signs: **complex** charges.

$$A = \frac{1}{2}\sum_{a,b,c} N_{ab}^{c} \; n^a \; n^b \; \chi_{k+2-c}$$

$$M = \pm\frac{1}{2}\sum_{a,b} N_{bb}^{a} \; n^b \; \hat{\chi}_{k+2-a} \left[= \sum_{ab} n^b\, \hat{\chi}_a\, \mathcal{Y}_{(k+2)b}^{a}\right]$$

- The $\mathcal{Y}$ **tensor** extends the Cardy ansatz to $\mathcal{M}$.

  (Pradisi, A.S. and Stanev, 1995)
  (Huiszoon, Schellekens and Sousa, 1999)

- Defined by **analogy** with $\mathcal{N}$, with $S \to P$:

$$\mathcal{Y}_{jk}{}^{i} = \sum_{l} \frac{P_{jl}\, S_{kl}}{S_{1l}}\, P^{\dagger}{}_{il}$$

- **(Signed) integer**, satisfies the **fusion algebra**:

$$\mathcal{Y}^{i}\; \mathcal{Y}^{j} = \sum_{k} \mathcal{N}^{ij}{}_{k}\; \mathcal{Y}^{k}$$

Using $\mathcal{M}$ to obtain $\mathcal{A}$

- In **non-diagonal** (non $\mathcal{C}$-conjugate) models, **no** one-to-one correspondence between boundaries and characters, **not all** bulk sectors allowed in tube. The Cardy ansatz does not apply. How can one **classify boundary operators**?

- One can build $\mathcal{A}$ **systematically**, starting from $\mathcal{M}$ and turning on **one charge at a time**. $\mathcal{M}$, **linear** in the charges, allows to derive the general solution by **superposition**.

- Basic structure:

$$\mathcal{A} = \frac{1}{2} \sum_{i\alpha\beta} \mathcal{A}^i_{\alpha\beta} \, n^\alpha n^\beta \chi_i$$

$$\mathcal{M} = \frac{1}{2} \sum_{i\alpha} \mathcal{M}^i_\alpha \, n^\alpha \chi_i$$

$$\mathcal{A}^i_{\alpha\alpha} = \mathcal{M}^i_\alpha \quad \text{(modulo 2)} \quad |\mathcal{M}| \leq \mathcal{A}$$

- I will now illustrate the method in a simple case: in general, **Diophantine equations** whose solutions are **small** (signed) integers.

The A_3 model "one charge at a time"

- **Start** from $\mathcal{K}$ and $\tilde{\mathcal{K}}$:

$$\begin{aligned}\mathcal{K} &= \frac{1}{2}(\chi_1 - \chi_2 + \chi_3)\\ \tilde{\mathcal{K}} &= \frac{1}{4}\Big((2-\sqrt{2})\chi_1 + (2+\sqrt{2})\chi_3\Big)\end{aligned}$$

- For this model:

$$S = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \quad P = \begin{pmatrix} \sin(\frac{\pi}{8}) & 0 & \cos(\frac{\pi}{8}) \\ 0 & 1 & 0 \\ \cos(\frac{\pi}{8}) & 0 & -\sin(\frac{\pi}{8}) \end{pmatrix}$$

- Since $\tilde{\mathcal{K}}$ **does not** contain χ_2, the same is true for $\tilde{\mathcal{M}}$ and $\mathcal{M}$, and:

$$\mathcal{M} = \frac{1}{2}(\epsilon_1\,\chi_1 + \epsilon_3\,\chi_3)$$

$$2\tilde{\mathcal{M}} = \left[\epsilon_1 \sin(\frac{\pi}{8}) + \epsilon_3 \cos(\frac{\pi}{8})\right]\chi_1 + \left[\epsilon_1 \cos(\frac{\pi}{8}) - \epsilon_3 \sin(\frac{\pi}{8})\right]\chi_3$$

- The ϵ_i are **small signed integers**, the **coefficients** of a given n_α in $\mathcal{M}$.

- Now from $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{M}}$ we can obtain $\tilde{\mathcal{A}}$ (up to an **arbitrary** reflection coefficient, γ, for χ_2) and, after a modular S transformation, $\mathcal{A}$.

"One charge at a time" $\left[(c,s)=(\cos,\sin)(\frac{\pi}{8})\right]$

$$\tilde{\mathcal{A}} = \frac{(\epsilon_1 s+\epsilon_3 c)^2}{2-\sqrt{2}}\chi_1 + \gamma^2\chi_2 + \frac{(\epsilon_1 c-\epsilon_3 s)^2}{2+\sqrt{2}}\chi_3$$

$$\mathcal{A} = \frac{1}{4}(\epsilon_1^2+3\epsilon_3^2+2\epsilon_1\epsilon_3+2\sqrt{2}\gamma^2)\chi_1+\epsilon_3(\epsilon_1+\epsilon_3)\chi_2 + \frac{1}{4}(\epsilon_1^2+3\epsilon_3^2+2\epsilon_1\epsilon_3-2\sqrt{2}\gamma^2)\chi_3$$

- **Comparing** $\mathcal{A}$ and $\mathcal{M}$, one obtains (mod. 2):

$$\epsilon_3(\epsilon_1+\epsilon_3)=0$$
$$\epsilon_1^2+3\epsilon_3^2+2\epsilon_1\epsilon_3+2\sqrt{2}\gamma^2=2|\epsilon_1|$$
$$\epsilon_1^2+3\epsilon_3^2+2\epsilon_1\epsilon_3-2\sqrt{2}\gamma^2=2|\epsilon_3|$$

- **Two solutions**: $(\epsilon_1,\epsilon_3,\gamma) = (1,0,\frac{1}{2\sqrt{2}})$ and $(-1,1,0)$

- The first is actually **a pair** (n_1,n_3) ($\gamma^2=\frac{(n_1-n_3)^2}{2\sqrt{2}}$):

$$\mathcal{M}=\frac{1}{2}(n_1+n_3)\chi_1$$
$$\mathcal{A} = \frac{1}{4}((n_1+n_3)^2+2\sqrt{2}\gamma^2)\chi_1+\frac{1}{4}((n_1+n_3)^2-2\sqrt{2}\gamma^2)\chi_3$$

- **General** $\mathcal{A}$ ($\epsilon_1\to n_1+n_3-n_2$, $\epsilon_2\to n_2$):

$$\mathcal{A}=\frac{1}{2}\left[(n_1^2+n_2^2+n_3^2)\chi_1+2n_2(n_1+n_3)\chi_2+(n_2^2+2n_1n_3)\chi_3\right]$$

The D_{odd} models

- One can repeat this procedure for all ADE models. In the D_{odd} series, we originally **missed** some boundary conditions.

(Pradisi, A.S., Stanev, 1995)

- **Genuinely off-diagonal** models: I will discuss the simplest, the **real** descendants of the D_5 model ($k = 6$):

$$T = |\chi_1|^2+|\chi_3|^2+|\chi_5|^2+|\chi_7|^2+|\chi_4|^2+(\chi_2\bar{\chi}_6+h.c.)$$
$$K^r = \frac{1}{2}(\chi_1+\chi_3+\chi_5+\chi_7-\chi_4)$$
$$K^c = \frac{1}{2}(\chi_1+\chi_3+\chi_5+\chi_7+\chi_4)$$

- One would expect **five** charges, but the method (looking for $|\epsilon_i| \leq 1$) gives only **four**:

$$A = \frac{1}{2}\Big[\chi_1(n_1^2+n_2^2+n_3^2+n_4^2)+2n_1n_2(\chi_2+\chi_6)$$
$$+ \chi_3(n_1^2+2n_1n_3+2n_1n_4+2n_3n_4)+\chi_4(2n_2n_3+2n_2n_4)$$
$$+ \chi_5(n_1^2+n_3^2+n_4^2+2n_1n_3+2n_1n_4)+\chi_7(n_1^2+n_2^2+2n_3n_4)\Big]$$

The D_{odd} models, again

(Pradisi, A.S. and Stanev, 1996)

- Actually, one can also **solve** $B_i^a \, B_j^a = \sum_k X_{ij}{}^k \, B_k^a$.

- The **structure constants** X for this class of models can deduced from some previous work of Petkova and Zuber (1995), and are **different** for **diagonal** (A) and **non-diagonal** (D_{odd}) models!

- For the **diagonal** A_7 model: **seven** charges.

- For this model: **five charges**, not four !

- Two related phenomena:

1. As in the simple 10D case, **some** boundary states are combinations of pairs of **diagonal** ones, but there is a **splitting**.

2. The corresponding boundary operators fuse with **multiplicities**, that we had excluded, thus **loosing** one solution.

- In other words: an **extension** inherited from the **boundary** algebra!

Extension of the boundary symmetry

- The **complete** $\mathcal{A}$ is:

$$A = \frac{1}{2}\Big\{\chi_1(n_1^2 + n_2^2 + n_3^2 + n_4^2 + \mathbf{n_5^2})$$
$$+ 2[n_1 n_2 + (n_1 + n_3 + n_4)\mathbf{n_5}](\chi_2 + \chi_6)$$
$$+ \chi_3(n_1^2 + 2n_1 n_3 + 2n_1 n_4 + 2n_3 n_4 + 2n_2 \mathbf{n_5} + \boxed{2\mathbf{n_5^2}})$$
$$+ \chi_4(\boxed{4n_1 \mathbf{n_5}} + 2n_2 n_3 + 2n_3 \mathbf{n_5} + 2n_2 n_4 + 2n_4 \mathbf{n_5})$$
$$+ \chi_5(n_1^2 + n_3^2 + n_4^2 + \boxed{2\mathbf{n_5^2}} + 2n_1 n_3 + 2n_1 n_4 + 2n_2 \mathbf{n_5})$$
$$+ \chi_7(n_1^2 + n_2^2 + \mathbf{n_5^2} + 2n_3 n_4)\Big\}$$

- "Boxed" terms signal **multiple** boundary operators.

- This phenomenon is **standard** in D_{even} models, where the **extension** is inherited from the **bulk**.

- For D_{even}: the **bulk** extension is due to **simple currents** of **integer** dimension.

- D_{odd} models have **simple currents** of **1/2-integer** dimension (here χ_7, with $h = 3/2$). They can not extend the **bulk** algebra, but **can** and **do** extend the **boundary** algebra!

Polynomial equations for $\mathcal{A}$

- The annulus with **five** charges has a special property: **completeness**.

- The matrices $\mathcal{A}$ satisfy a set of **polynomial equations**:

$$\begin{aligned}\sum_{\gamma} \mathcal{A}_{i\alpha}{}^{\gamma}\ \mathcal{A}_{j_\gamma}{}^{\beta} &= \sum_{k} \mathcal{N}_{ij}{}^{k} \mathcal{A}_{k\alpha}{}^{\beta} \\ \sum_{i} \mathcal{A}_{i\alpha}{}^{\beta}\ \mathcal{A}^{i}{}_{\mu}{}^{\nu} &= \sum_{i} \mathcal{A}_{i\alpha}{}^{\nu}\ \mathcal{A}^{i}{}_{\mu}{}^{\beta}\end{aligned}$$

1. The **first** reflects the **completeness** of the boundary operators. $\mathcal{A}_{i\alpha}{}^{\gamma}$ **counts** the **three-point** couplings $< \alpha|\phi_i|\gamma^c >$, and can **compute** $< \alpha|\phi_i\phi_j|\gamma^c >$in two ways:

$$\begin{aligned}< \alpha|\phi_i\phi_j|\beta^c > &= \sum_{k} \mathcal{N}_{ij}{}^{k} < \alpha|\phi_k|\beta^c > \\ < \alpha|\phi_i\phi_j|\beta^c > &= \sum_{\gamma} < \alpha|\phi_i|\gamma^c >< \gamma|\phi_j|\beta^c >\end{aligned}$$

2. The **second** reflects the **annulus constraint**.

- $\mathcal{A}_{i\alpha}{}^{\beta}$: **(positive) integer matrix representations of the fusion algebra**, whose **dimensions** are determined by the sectors allowed in the tube.

ADE and $SU(2)$ boundary operators

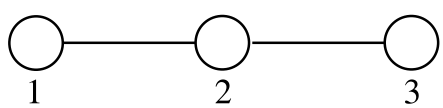
(Behrend, Pearce, Petkova and Zuber, 1998)

- The classification problem is solved in principle by the **single-charge** algorithm (**fully** solved, for $SU(2)$ WZW), while the polynomial equations were used to **verify** the **completeness**. In $SU(2)$ WZW (and minimal) models, one can **solve** directly the equations, relating them to ADE Dynkin diagrams.

- **Basic idea**: need only the **generators** of the **fusion ring** (here the $\mathcal{A}$ matrix for **isospin 1/2**).

- For the A_3 model, we have seen that :

$$\mathcal{A}^2 = \mathcal{A}^{(I=1/2)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



- This is the **adjacency matrix** of A_3, and the correspondence works for **ADE modular invariants**.

Consequences of the spectral decomposition

(Stanev, DESY 98, A.S. and Stanev, to appear)
(Behrend, Pearce, Petkova and Zuber, 1999)

- **Projectors** on states allowed in the tube: $\Pi_i = \delta_{ii^c}$

- **Define (for SL(2))**

$$R^{i\alpha} = B^{i\alpha} \sqrt{S_{1i}}\ \Pi^i$$

- The the spectral decomposition becomes

$$A_j^{\alpha\beta} = \sum_i \frac{S_{ji}^\dagger}{S_{1i}}\ R^{i\alpha}\, R^{i\beta}$$

- The polynomial equations imply that the **rectangular** matrices R are (left and right) **orthogonal**:

$$\sum_\alpha R^{i\alpha}\ {R^j}_\alpha \quad = \quad \delta^{ij}\ \Pi^i$$
$$\sum_i R^{i\alpha}\ R_{i\beta} \quad = \quad \delta^\alpha_\beta$$

- The **structure constants** X are also determined in terms of the R.

Consequences of the spectral decomposition

(Stanev, DESY 98, A.S. and Stanev, to appear)

- One can prove **trace formulae**:

$$tr(A_{i_1} A_{i_2} \dots A_{i_n}) = tr(N_{i_1} N_{i_2} \dots N_{i_n} Z)$$

- Finally, one can prove additional **polynomial equations** for $\mathcal{K}$ and $\mathcal{M}$:

$$\begin{aligned} \sum_i M_i^\alpha \; M^{i\beta} &= \sum_i A_i^{\alpha\beta} \; K^i \\ \sum_\beta A_i^{\alpha\beta} \; M_{j\beta} &= \sum_\ell \; Y_{ij}{}^\ell \; M_\ell{}^\alpha \\ \sum_\beta M_i^\beta \; M_{j\beta} &= \sum_\ell Y^\ell{}_{ij} \; K_\ell \end{aligned}$$

- **Reverse procedure**: from open to closed?

- $\mathcal{K}$ and $\mathcal{A}$ essentially determined !

Conclusion

- I have reviewed how the idea of **open descendants** or **orientifolds**, stimulated by the structure of the two originally known open-string models, has produced a number of interesting consequences:

1. New **non supersymmetric** 10D models, with(out) **tachyons** and large classes of corresponding **compactifications**.

2. **New phenomena** in 6D vacua, where transitions related to **tensionless strings** are nicely captured by the corresponding low-energy **supergravity**.

3. New classes of **chiral** 6D and 4D models, that are changing our perspective on low-energy physics: **brane world**.

4. New possibilities for **supersymmetry breaking**. In particular, a scenario, demanded by the consistency of some **chiral** models, (**brane supersymmetry breaking**) where supersymmetry, broken at **string scale** in our world, is exact at tree level for the **bulk** gravity.

Conclusion

5. A new (signed) integer-valued **tensor** in 2D CFT, built by analogy with the **fusion** tensor $\mathcal{N}^i{}_{jk}$, but involving the P **matrix**, that **completes** the Cardy ansatz to $\mathcal{K}$ and $\mathcal{M}$: $\mathcal{Y}^i{}_{jk}$.

6. An **algorithm** to **classify** the **boundary operators** in 2D RCFT. Or, in string language, to build (in principle) general open descendants.

6. **Polynomial equations** for $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$, that reduce the problem of **classifying boundary operators** to the problem of studying the **positive integer** matrix representations of the fusion algebra.