Research and main results of Luigi Ambrosio

1. Mumford-Shah problem, image segmentation and free discontinuity problems.

Between 1988 and 1995 Ambrosio has mainly been studying a class variational problems involving the minimization of volume and surface energies. The model problem is the minimization of the Mumford–Shah functional

\[
\int_{\Omega \setminus K} |\nabla u|^2 + \alpha (u - g)^2 \, dx + \beta H^{n-1}(\Omega \cap K)
\]

where \( \Omega \subset \mathbb{R}^n \) is an open set, \( H^{n-1} \) is the Hausdorff \( (n - 1) \)-dimensional measure in \( \mathbb{R}^n \) and \( g \in L^\infty(\Omega) \). The pair \((u, K)\) is admissible for the problem if \( K \) is a closed set and \( u \in C^1(\Omega \setminus K) \).

Mumford and Shah proposed this model in the framework of a variational approach to image segmentation, but this model (due to the strong links with Griffith’s theory) has received also a lot of attention in fracture mechanics, e.g. for building a rigorous theory of quasi-static evolution of cracks.

Since \( u \) may be discontinuous across \( K \), the analysis of (1) leads in a natural way to weak formulations in spaces of discontinuous functions in \( \Omega \). This is done in a paper by Ambrosio and De Giorgi [9], where a space \( SBV \) of “special” functions of bounded variation is proposed. In [10] and [11] Ambrosio builds a general existence theory, providing weak solutions for variational problems, modeled on (1), with general energy densities, depending on the normal \( \nu_K \) to \( K \) and on the jump of \( u \) across \( K \):

\[
\int_{\Omega \setminus K} f(x, u, \nabla u) \, dx + \int_{\Omega \cap K} \varphi(u^+, u^-, \nu_K) \, dH^{n-1}.
\]

Later on the proof of the main compactness theorem in [10] has been simplified and put in a geometric context ([37], [85]) and the theory has been extended to cover the typical energy densities \( f \) which occur in non-linear elasticity theory, quasi-convex with respect to \( \nabla u \) [34].

The regularity theory for solutions of (1) has also been studied by many authors (Ambrosio, David, De Giorgi, Carriero, Dal Maso, Fusco, Morel, Pallara, Solimini, Semmes, Bonnet). The Mumford–Shah conjecture, still open, says that \( K \) in the two dimensional case any optimal set \( K \) has (locally) finitely many singular points. In the years 1995-97 there was some relevant progress on this problem: Bonnet proved the conjecture under the a priori assumption that \( K \) has finitely many connected components and David, and Ambrosio, Fusco and Pallara in [39], [40] proved without a priori assumptions that any optimal set \( K \) is \( C^{1,\alpha} \) out of a closed singular set \( \Sigma \subset K \), with \( H^1(\Sigma) = 0 \). The result by Ambrosio, Fusco and Pallara holds in any dimension (in this case \( H^{n-1}(\Sigma) = 0 \) and later it was proved that the \( H - \text{dim}(\Sigma) < n - 1 \)) and it is still the only regularity result in this subject known to be true in any number of dimensions.

In 1990 Ambrosio and Tortorelli proved in [16] (see also [23]) that the Mumford–Shah functional can be approximated, in the sense of \( \Gamma \)-convergence, by the family of functionals

\[
\int_{\Omega} v^2 |\nabla u|^2 + \alpha (u - g)^2 \, dx + \frac{\beta}{2} \int_{\Omega} \epsilon |\nabla v|^2 + \frac{(1 - v)^2}{\epsilon} \, dx.
\]

This approximation raised a lot of interest, from the point of view of numerical simulations (as the functionals in (2) are easier to handle) and from the theoretical viewpoint (in view of the justification of continuum models as a limit of discrete ones) and the papers [16] and [23] are by now a standard reference in this subject.

The theory developed by Ambrosio and collaborators is summarized in the book [66], that is also a reference book on the theory of \( BV \) functions. The book contains several new results about \( BV \)
functions, for instance the chain rule $D(f \circ u)$ with $u \in BV$ vector-valued and $f$ Lipschitz (this comes from [14], even the Sobolev case was not known).

2. Geometric evolution problems.

In the last 20 years there has been an intensive research activity in evolution problems of geometric type, whose main model can be considered the flow by the mean curvature, (heuristically the gradient flow of the area functional). One of the most popular methods, introduced by Osher and Sethian, is based on the representation of the moving surface as the level set of a function solving an auxiliary PDE. This method fits very well into the theory of viscosity solutions. In [38] Ambrosio and Soner extend this theory to flows of surfaces of any codimension. In this case a crucial role is the analysis of the relation between the geometry of the manifold and the derivatives of the squared distance function from the manifold. This theme is further analyzed in [48].

Another method for studying flows by mean curvature, initiated by Brakke in 70’s, is based on the analysis of the surface measures carried by the moving manifold. This method has strong links with the Allard–Almgren theory of varifolds. In codimension greater than 1 there is no obvious way to relate the family of level sets to a varifold, and this was the main obstruction in the extension to codimension greater than 1 (i.e. to vector-valued maps) of Ilmanen’s convergence proof of the reaction-diffusion equation

$u_t = \Delta u - \frac{u(1-u^2)}{\epsilon^2}$

(3) to a flow by mean curvature in the sense of Brakke. In [53] Ambrosio and Soner introduce a more general theory of varifolds (based on a larger Grassmannian) and use it to prove a convergence result in codimension 2, when the equation (3) is of Ginzburg–Landau type, assuming a priori the existence of a density lower bound on the limiting measures. Bethuel, Orlandi and Smets, coupling the measure-theoretic analysis of [53] with hard PDE estimates have obtained a general convergence result, still in codimension 2.

3. Analysis in metric spaces.

In 1990, motivated by the study of the study of the asymptotic limit (in the sense of Γ-convergence) of the problems

$$\min \left\{ \int_\Omega |\nabla u|^2 + \frac{1}{\epsilon} F(u) + \frac{1}{\epsilon^2} W^2(u) \right\}$$

when $u$ is a vector-valued map and there is no special restriction on $\{W = 0\}$, Ambrosio introduced in [18] the concept of $BV$ map between a domain $\Omega \subset R^n$ and a general metric space $(E, d)$. In the Sobolev case (a particular case) the definition reads as follows: $u \in W^{1,p}(\Omega, E)$ if there exists a function $g \in L^p(\Omega)$ with the property that $\phi \circ u \in W^{1,p}(\Omega)$ for any real-valued and 1-Lipschitz map $\phi : E \to R$ and $|\nabla(\phi \circ u)| \leq g$ a.e. in $\Omega$. This definition is clearly fully intrinsic and consistent with others, more restrictive, which typically require coordinates or the use of an isometric embedding of $E$ into an Euclidean space. Some years later this concept was independently discovered and popularized by Reshetnyak. Eventually, in 1996, Hajlasz gave the complementary definition of Sobolev map between a metric space and an Euclidean space. The two definitions can be merged, giving natural concepts of Sobolev and BV maps between arbitrary metric spaces, as discussed in the book [70].

The theory of $BV$ functions with values in metric spaces plays also a fundamental technical role in the paper [60] (see also [59]), where Ambrosio and Kirchheim extend the Federer–Fleming theory of $m$-currents to any metric space following an approach suggested by De Giorgi, based on the
formal duality with a class of suitable Lipschitz $m$-forms (and analogous to [18]). The results of [60] have been a great surprise for the Geometric Measure Theory community, since all proofs of the typical results of the Federer–Fleming theory (closure for rectifiable currents with mass bounds on the boundary, boundary rectifiability theorem, existence of weak solutions for Plateau’s problem) were heavily using the Euclidean structure of the ambient space, for instance the Besicovitch differentiation theorem. Roughly speaking the results of [60] extend to $m$-dimensional parametric surfaces the following well known 1-dimensional fact: geodesics (understood as length minimizing curves) exist in any locally compact metric space, provided the metric space contains curves of finite length. The theory developed in [60] provides new results even in the Euclidean case (for instance rectifiability criteria based on slices or on Lipschitz projections on $\mathbb{R}^{m+1}$) and is very well adapted to the Gromov–Hausdorff theory of convergence of metric spaces. Using these tools Ambrosio and Kirchheim show existence results for Plateau’s problem even when the ambient space is not locally compact, for instance in general classes of infinite dimensional Banach spaces (even the Hilbert case was not known).

In [65] Ambrosio has proved a general version of De Giorgi’s rectifiability theorem in Ahlfors regular metric measure spaces for which an abstract version of the Poincaré inequality holds (two years later the result has been improved in [72]). The main difficulty in proving the result was the absence of doubling properties of the perimeter measure and of a Besicovitch covering theorem in general metric spaces. This result opened the way to more detailed investigations of the rectifiability problem in Carnot groups (by Franchi, Serapioni and Serra Cassano), where more structure is available.

4. Optimal transport theory.

Let $\Omega \subset \mathbb{R}^n$ be a convex set and let $\mu, \nu$ be probability measures in $\Omega$. The problem of optimal transportation

$$\inf\left\{ \int_{\Omega} c(x, \psi(x)) \, dx : \psi_{\#}\mu = \nu \right\},$$

raised by Monge in 1781, has found in recent years an enormous attention, due to its connections with Calculus of Variations, Fluid Mechanics, Probability, Economics and other fields. The existence of optimal transport maps is a delicate problem, due to the non-convexity of the constraint $\psi_{\#}\mu = \nu$.

The case when $c$ is a strictly convex function of the Euclidean distance has been studied, at various levels of generality, by many authors (Brenier, Caffarelli, Gangbo–McCann, Smith–Knott, Rachev–Ruschendorf) and by now it can be considered completely understood. The other cases are more delicate because the necessary and sufficient optimality conditions (coming from the dual formulation of Kantorovich’s problem) fail to give enough informations about the direction of transport and/or the length of transportation.

For instance in the case $c(x, y) = |x - y|$, corresponding to Monge’s original problem, there are informations on the direction of transportation but not on the length of transportation. The first attempt to bypass this difficulty came in 1978 with the work of Sudakov, who claimed to have a solution for any distance cost function induced by a norm. Sudakov’s approach is based on a clever decomposition of the space $\Omega$ in affine regions with variable dimension where the Kantorovich potential associated to the transport problem is an affine function. His strategy is to solve the transport problem in any of these regions, eventually getting an optimal transport map just by gluing all these transport maps. An essential ingredient is his statement that if $\mu$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, then the conditional measures $\mu_C$ induced by the decomposition are absolutely continuous with respect to the Lebesgue measure on $C$ (of the correct dimension). But it turns out that, as pointed out in [69], when $n > 2$ the property claimed
by Sudakov is not true even for the decomposition in segments: in [82] Ambrosio, Kirchheim and Pratelli exhibit a remarkable example (based on an improvement of a construction suggested by Alberti, Kirchheim and Preiss) of a Borel family of pairwise disjoint open segments \{l_i: i \in I\} and points \(x_i \in l_i\) in \(\mathbb{R}^3\) such that the Borel set \(\{x_i: i \in I\}\) covers \(\mathcal{L}^3\)-almost all of the unit cube \(Q\). Therefore, by applying the disintegration theorem with this family of segments to \(\mu = \chi_Q \mathcal{L}^3\), we find that the conditional measures \(\mu_C\) are actually Dirac masses!

In more recent years the problem of existence of optimal transport maps in the case cost=distance had been attacked by new methods, somehow still related to Sudakov’s one but using some regularity property of the decomposition, a property that now we know to be essential. The first progress came with the work of Evans–Gangbo, where existence of optimal transport maps is proven assuming absolute continuity of \(\mu\) and \(\nu\) with respect to \(\mathcal{L}^n\) and smoothness of the densities. The first existence results for general absolutely continuous measures \(\mu, \nu\) with compact support have been independently obtained by Caffarelli, Feldman and McCann and by Trudinger and Wang. In [69] Ambrosio shows that the absolute continuity of \(\mu\) only is sufficient to provide existence and a class of examples built in [75] shows that this assumption is basically sharp (this is not true for strictly convex cost functions).

Despite these progresses the existence of optimal transport maps for the case \(c(x, y) = \|x - y\|\), with \(\|\cdot\|\) generic norm in \(\mathbb{R}^n\), is still an open problem: indeed, if either the norm is not strictly convex or it is not \(C^2\) then the information on the direction of transportation or on the regularity of the decomposition could be lost. The above mentioned paper [82] contains an existence result for the case \(\|x\| = \|x\|_\infty\), based on an asymptotic development by \(\Gamma\)-convergence and on the approximation by the cost functions

\[
c_\epsilon(x, y) = \|x - y\|_\infty + \epsilon |x - y| + \epsilon^2 |x - y| \ln |x - y|
\]

(roughly speaking the first term rules out the ambiguity on the direction of transportation and the second one rules the ambiguity on its length).

The paper [79] with Rigot, devoted to optimal transport maps in the Heisenberg group, is the first general existence theorem in a genuine non-Euclidean (i.e. not Riemannian) context.

Finally, the book [84] is devoted to a general theory of gradient flows in metric spaces. Under geometric convexity conditions of the energy along constant speed geodesics and along suitable classes of generalized geodesics, error estimates, convergence of the Euler implicit time discretization scheme and uniqueness of gradient flows are proved. The theory is built having in mind as a model the space of probability measures in \(\mathbb{R}^n\) having finite second order moments, endowed with the Wasserstein distance. This line of investigation was initiated by the work of McCann (displacement convexity), Benamou–Brenier (the time-dependent optimal transportation problem), Jordan–Kinderlehrer–Otto (the Fokker–Planck equation viewed as a gradient flow) and Otto (the porous medium equation). The results of [84] cover all the previously known examples (and the error estimates are completely new), with a formalism which avoids the restriction to absolutely continuous measures. The monograph [84] is now, besides Villani’s book, one of the basic references in the theory of optimal mass transport.

5. Transport equation, Cauchy problem and conservation laws.

The extension of the Lions–DiPerna theory to \(BV\) vector fields has been an open problem since the appearence of the Lions–DiPerna paper in 1989. In more recent years there have been several contributions to this subject, by P.L. Lions, Bouchut, Colombini–Lerner, Hauray. In [780] Ambrosio has been able to achieve this extension, assuming (besides the \(BV\) regularity) only that the distributional divergence is absolutely continuous and has bounded density with respect to the Lebesgue measure. The proof is based on careful dyadic decomposition and regularization arguments and on the theory of Young measures.
This result opens the possibility to solve some non-linear PDE’s with $BV$ data using the method of characteristics. In [81] Ambrosio and De Lellis use this method to give a general existence result for the Cauchy problem relative to a class of systems of conservation laws of the form

$$ u_t + \sum_{i=1}^n \partial_i f_i(|u|u) = 0. \tag{4} $$

Although the nonlinearity in (4) has a very special form, this is the first general existence result for systems with more than 2 space variables and more than 2 equations. Later on, the result has been improved in [83], allowing for more general initial data and including some uniqueness results.

6. The most recent contributions.

In the last few years Ambrosio, while continuing to work in the area listed above, focussed on two main new themes: the development of Geometric Measure Theory in infinite-dimensional spaces and the study of metric measure spaces with tools borrowed from the theory of optimal mass transportation. In the first direction, the papers [121], [134] and [150] provide infinite-dimensional analogues of classical finite-dimensional results: the main new ingredient is the use of semigroup analogues for concepts like approximate continuity or spherical density which have no analogue or become problematic in infinite-dimensional spaces, while [146] provides a new approach to the existence of minimal surfaces in infinite-dimensional spaces. In the other direction, in a series of new papers with N.Gigli and G.Savaré [137], [140], [149] he developed a completely new approach to the theory of Sobolev and BV spaces on metric measure spaces which is suitable both for finite and infinite dimensional ones. Also, in [138] he introduced a new notion of Riemannian Ricci bounds from below which enforces the axiomatization of Lott, Sturm and Villani while retaining good stability properties with respect to the Gromov-Hausdorff convergence. In this respect, this notion seems to be more appropriate for the description of the closure of the class of Riemannian manifolds with uniform lower bounds on Ricci curvature.
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