RESEARCH DESCRIPTION

During the year 2017 I plan to work on two main themes.

Motivic classes of classifying stacks for connected algebraic groups. Let $k$ be a field. The Grothendieck ring of varieties $K_0(\text{Var}_k)$ was first defined by Grothendieck in 1964 in a letter to Serre. Its main application so far is Kontsevich’s theory of motivic integration: see for example [Loo00].

Variants of this, that contain classes for all algebraic stacks of finite type over $k$ with affine stabilizers, have been introduced by several authors: see [BD07], [Ekeb], [Joy07], [Toe05]. In the present paper we use the version due to Ekedahl, which we denote by $K_0(\text{Stack}_k)$; it has the merit of being universal, so it maps to all the other versions.

By definition, every algebraic stack $\mathcal{X}$ of finite type over $k$ with affine stabilizers has a class $\{\mathcal{X}\}$ in $K_0(\text{Stack}_k)$. In particular, given an affine group scheme of finite type $G$ over $k$, we obtain a class $\{BG\}$ for the classifying stack $BG$ in $K_0(\text{Stack}_k)$. The problem of computing $\{BG\}$ is very interesting; it is morally related with the problem of stable rationality of fields of invariants for generically free representations, although no direct implication is known (see the discussion in [Ekea, § 6]).

The case of a finite group is thoroughly discussed in [Ekea]; in many cases $\{BG\} = 1$, although there are examples of nilpotent finite groups for which this fails.

The case when $G$ is connected is also very interesting. Recall that an algebraic group is special if every $G$-torsor is Zariski-locally trivial; $\text{GL}_n$, $\text{SL}_n$ and $\text{Sp}_n$ are all special. If $P \to S$ is a $G$-torsor and $G$ is special, then we have $\{P\} = \{G\}\{S\}$ (this is immediate when $S$ is a scheme, and it was shown by Ekedahl when $S$ is an algebraic stack). In particular, applying this to the universal torsor $\text{Spec}\ k \to BG$ we get the formula $\{BG\} = \{G\}^{-1}$ for special groups.

It is somewhat surprising that the equality $\{BG\} = \{G\}^{-1}$ holds for several $G$ that are not special. These are the known cases.

1. $G = \text{PGL}_2$ and $\text{PGL}_3$ (by D. Bergh, [Ber16]).
2. $G = \text{SO}_n$ (by A. Dhillon and M. Young in [DY16] when $n$ is odd, by M. Talpo and myself in [TV] for the general case).
3. $G = \text{Spin}_n$ for $n \leq 8$ and when $G$ is the simply connected form of $\text{G}_2$ (by R. Pirisi and M. Talpo, unpublished).

This might be related with the fact that quotient spaces of generically free representations of connected algebraic groups tend to be stably rational; in fact, no examples are known in which they are not rational (see [Böh] for a survey of the known results in this direction).

I have started investigating the case $G = \text{PGL}_n$. The only known technique for computing $\{BG\}$ is to take a representation of $G$ and stratify it, so that the classes of the strata can be computed. The obvious representation to use for $\text{PGL}_n$ is the adjoint representation; this idea has been used for studying the Chow ring of $\mathcal{B}\text{PGL}_n$ (see [Vez00, Vis07]). The orbit structure is much complex that that, for
We define a stack \( X \) when \( \mathcal{A} \) is a quotient stack \([U/G]\), where \( U \) is a smooth scheme of finite type over a field \( k \) and \( G \) is an affine algebraic group on \( k \), we obtain a Chow ring \( A^*_G(U) = A^*(\mathcal{X}) \), which only depends on \( \mathcal{X} \) and not on the presentation of \( \mathcal{X} \) as a quotient stack. If \( \mathcal{X} \) is Deligne–Mumford, or, equivalently, the action of \( G \) on \( U \) has finite reduced stabilizers, then \( A^*(\mathcal{X}) \otimes \mathbb{Q} \) coincides with the rational Chow ring of \( \mathcal{X} \), which had been earlier studied by several authors ([Mum83, Gil84, Vis89].

The ring \( A^*(\mathcal{X}) \) is usually much harder to compute than \( A^*(\mathcal{Z}) \otimes \mathbb{Q} \); for example, consider the moduli stack \( \mathcal{M}_g \) of smooth curves of genus \( g \), with \( g \geq 2 \); the ring \( A^*(\mathcal{M}_g) \) has been computed only for \( g = 2 \) [Vis98] (notice that in this case \( A^*(\mathcal{M}_2) \) coincides with the rational Chow ring of \( \mathcal{Z} \)).

In all these calculations the essential point is the determination of the Chow ring of certain stacks of hypersurfaces. More precisely, let \( n \) and \( d \) be positive integers. We define a stack \( \mathcal{Y}_{n,d} \) as follows: an object of \( \mathcal{Y}_{n,d} \) over a \( k \)-scheme \( S \) consists of a vector bundle \( F \) of rank \( n \), and a Cartier divisor \( X \subseteq \mathbb{P}(F) \) whose restriction to every fiber is a hypersurface of degree \( d \).

An alternate description of \( \mathcal{Y}_{n,d} \) is as follows. Denote by \( W_{n,d} \) the vector space of homogeneous polynomials of degree \( d \) in \( n \) variables, with its natural action of \( \text{GL}_n \). Set \( P_{n,d} = \mathbb{P}(W_{n,d}) \); so \( P_{n,d} \) is the projective space of hypersurfaces of degree \( d \) in \( \mathbb{P}^{n-1} \). If \( Z \subseteq P_{n,d} \) is the discriminant locus, we have

\[
\mathcal{Y}_{n,d} = [(P_{n,d} \setminus Z)/\text{GL}_n].
\]

By standard facts of equivariant intersection theory, this gives a set of generators for the ring \( A^*(\mathcal{Y}_{n,d}) = A^*_{\text{GL}_n}(P_{n,d} \setminus Z) \), which are the Chern classes \( c_1, \ldots, c_n \) of the tautological representation of \( \text{GL}_n \), and \( h = c_1(O_{P_{n,d}}(1)) \). The relations among these generators \( c_1, \ldots, c_n, \) and \( h \) are obtained from the classes of the image of the pushforward \( A_{\text{GL}_n}(Z) \to A_{\text{GL}_n}(P_{n,d}) \).

A set of natural relations are obtained as follows. Let \( \tilde{Z} \subseteq P_{n,d} \times \mathbb{P}^{n-1} \) be the reduced subscheme consisting of pairs \((X,p)\), where \( X \) is a hypersurface of degree \( d \) in \( \mathbb{P}^{n-1} \), and \( p \) is a singular point of \( X \). Then \( \tilde{Z} \) is the image of \( Z \) in \( P_{n,d} \), hence every class in \( A^*_{\text{GL}_n}(\tilde{Z}) \) when pushed down to \( A^*_{\text{GL}_n}(P_{n,d}) \) gives a relation in \( A^*_{\text{GL}_n}(P_{n,d}) \). The push forward of the Chow ring \( A^*_{\text{GL}_n}(\tilde{Z}) \) is easily determined;
this gives certain relations $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[c_1, \ldots, c_n, h]$. When $d = 2$ or $n = 2$ it is proved in [EF08] that $\alpha_1, \ldots, \alpha_n$ generate the ideal of relations, so that

$$A^*(\mathcal{X}_{n,d}) = \mathbb{Z}[c_1, \ldots, c_n, h]/(\alpha_1, \ldots, \alpha_n).$$

In the general case, it is easy to see that the $\alpha_i$ generate the ideal of relation of the generators in $A^*(\mathcal{X}_{n,d}) \otimes \mathbb{Q}$.

In [FV16], Damiano Fulghesu and myself have computed the ideal of relations in the case $n = 3$, $d = 3$; it turns out that, besides the relations $\alpha_1$ and $\alpha_2$ above, one extra generator in degree 2.

Together with Fulghesu I plan to work on the next case, $A^*(\mathcal{X}_{3,4})$, the case of plane quartics. This is particularly interesting, because it would give as a byproduct a presentation of the Chow ring of non-hyperelliptic curves of genus 3.

We have a fairly clear idea of the geometry involved, and a strategy for the calculation; the details, however, are fairly complex.

References


